

# Note one , Math21A

J. Yin

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## 1 Introduction

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2. CA: Jenny Wang, Junior, Wang24@fas.harvard.edu.
3. Office Hours: T, Th 7pm-8pm, or by appointment.
4. Homework: The first part is due on Thursday and the next two are due on Tuesday (before the class).

## 2 Three dimensional Rectangular Coordinate System

Definition and Notations

1. Fixed point O(origin)
2.  $x, y, z$  axis (straight lines with direction)
3. Perpendicular to each other (12 right angles)
4. right hand rule
5. Three coordinate axis, three coordinate planes, eight octants

The point can be represented by a ordered triple  $(x, y, z)$

1. Which point is  $(1, 2, 3)$
2. Can we change the order of the procedure?

3. Is  $(1, 2, 3)$  equal to  $(3, 2, 1)$ ?

Examples:

1. What does the equation  $x = 3$  represent in space?

$$\{(x, y, z) : x = 3\} \quad (1)$$

2. What is the distance from point  $(1, 2, 3)$  to  $xy$ -plane?

### 3 Distance

Definition and Formula

1. The definition of Distance (in space?): Distance of two points is the length of the line segment between these two points.

2. Notations:

$$\text{dist}(P_1, P_2) \text{ or } |P_1P_2| \quad (2)$$

3. The distance of two points in plane. Pythagorean theorem. What is the distance from  $(0, 0)$  to  $(1, 2)$ ?

4. Calculate the distance between  $(0, 0, 0)$  and  $(1, 2, 3)$ .

5. Distance Formula:

$$\text{dist}((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad (3)$$

Examples:

1. Find an equation for the surface of the sphere with radius 3 and center  $(2, 1, 3)$ .

2. Find the center and radius of the following sphere:

$$x^2 + y^2 + z^2 + 2x + 4y + 6z = 0 \quad (4)$$

$$(x+?)^2 + (y+?)^2 + (z+?)^2 = ? \quad (5)$$

3. Find an equation for the set of all points whose distance from  $(0, 0, 1)$  is twice as the distance from  $(0, 0, -1)$ .

## 4 Vector

What is vector?

1. A quantity which has both magnitude and direction
2. Force, velocity, 20 feet, 2 o'clock
3. Displacement vector,  $\vec{AB}$ : initial point A, terminal Point B, magnitude is  $|AB|$ , direction is from A to B.
4. Equivalence.
5. zero vector has no direction

Vector addition

1. Triangle Law
2. Parallelogram Law
3. Example. (same magnitude, opposite direction?)

Scalar Multiplication

1. Definition of  $cv(c > 0)$ .
2. Definition of  $-v$
3. What is the meaning of  $u - v$ ?
4. Example.  $v + v = 2v$ ,  $v - v = 0$

Components

1.  $(x, y, z)$  is a point,  $\langle x, y, z \rangle$  is vector.
2. If  $A = (1, 0, 2)$  and  $B = (3, 0, 6)$ , then  $\vec{AB} = \langle ?, ?, ? \rangle$ .
3. The length of  $\langle x, y, z \rangle$  is equal to  $\sqrt{x^2 + y^2 + z^2}$
4. Vector addition and Scalar Multiplication
5. Unit vector,  $i, j, k$ ,  $\langle 2, 4, 3 \rangle = 2i + 4j + 3k$
6. Example 7 in textbook.

# Note Two , Math21A

J. Yin

April 29, 2009

## 1 Vector

What is vector?

1. A quantity which has both magnitude and direction
2. Force, velocity, 20 feet, 2 o'clock
3. Displacement vector,  $\vec{AB}$ : initial point A, terminal Point B, magnitude is  $|AB|$ , direction is from A to B.
4. Equivalence.
5. zero vector has no direction

Exercise:

1. Let  $A = (1, 2, 4)$ ,  $B = (2, 3, 2)$ , what is  $|AB|$ ,  $|BA|$ ,  $\vec{AB}$ ,  $\vec{BA}$ .
2. If  $\vec{OC} = \vec{AB}$ , the point C is  $(?, ?, ?)$

Components

1.  $(x, y, z)$  is a point,  $\langle x, y, z \rangle$  is vector.
2. The length of  $\langle x, y, z \rangle$  is equal to  $\sqrt{x^2 + y^2 + z^2}$
3. Vector addition and Scalar Multiplication. Check  $v+v = 2v$ ,  $v-v = 0$ .
4. Unit vector,  $i, j, k$ ,  $i = \langle 1, 0, 0 \rangle$ .  $\langle 2, 4, 3 \rangle = 2i + 4j + 3k$
5. Example 7 in textbook.

## 2 Dot product

1. The dot product  $\vec{v} \cdot \vec{w}$  of two vectors  $\vec{v}$  and  $\vec{w}$  is a real number and it is defined as follows.

- If  $\vec{v}$  and  $\vec{w}$  are two-dimensional vectors, say  $\vec{v} = \langle v_1, v_2 \rangle$  and  $\vec{w} = \langle w_1, w_2 \rangle$ , then their dot product is  $v_1w_1 + v_2w_2$ .
- If  $\vec{v}$  and  $\vec{w}$  are three-dimensional vectors, say  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  and  $\vec{w} = \langle w_1, w_2, w_3 \rangle$ , then their dot product is  $v_1w_1 + v_2w_2 + v_3w_3$ .

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

Is  $\vec{v} \cdot \vec{w}$  equal to  $|\vec{v}||\vec{w}|\cos\theta$ ?

2. Examples.

3. Properties.

4. Questions

- True or false: if  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are vectors of the same dimension, then  $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$ .
- What is the relationship between  $\vec{v} \cdot \vec{v}$  and  $|\vec{v}|$ ?
- Is  $\vec{v} \cdot \vec{w}$  equal to  $|\vec{v}||\vec{w}|\cos\theta$ ?
- Find the angle between  $\langle 1, 2, 1 \rangle$  and  $\langle 1, -1, 1 \rangle$ .

5. Projection

- Scalar projection of  $\vec{b}$  onto  $\vec{a}$ :

$$\text{Comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

- Vector projection of  $\vec{b}$  onto  $\vec{a}$ :

$$\text{Proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

- Find the vector projection of  $\langle 0, 0, 1 \rangle$  onto  $\langle 1, 2, 3 \rangle$ .
- If  $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ , we have?

### 3 Cross product

1. Definition.

The Cross product  $\vec{v} \times \vec{w}$  of two vectors  $\vec{v}$  and  $\vec{w}$  is a vector and it is defined as follows.

- The length  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$
- Direction  $\vec{n}$ , where  $\vec{n}$  is the unit vector orthogonal to both  $\vec{a}$  and  $\vec{b}$  so that  $\{a, b, n\}$  satisfies the right hand rule.

2. Exercises:

- $\vec{i} \times \vec{j}$
- $(\vec{i} + \vec{j}) \times (\vec{i} - \vec{j})$
- $\vec{v} \times \vec{w} + \vec{w} \times \vec{v}$
- $\vec{v} \cdot (\vec{v} \times \vec{w})$

3. The Cross product in Component Form Remember the determinant expression or remember this expression ( $\vec{u} = \langle u_1, u_2, u_3 \rangle, \vec{v} = \langle v_1, v_2, v_3 \rangle$ )

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \quad (1)$$

Exercises:

$$\langle 1, 2, 3 \rangle \times \langle 0, 1, -1 \rangle$$

4. Other properties

- Any two non-zero vectors  $\vec{v}$  and  $\vec{w}$  determine a parallelogram. What is the relationship between the area of the parallelogram and  $\vec{v} \times \vec{w}$ .
- If  $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$ , what can we say about the vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$ .
- If  $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$ , then  $v = w$ , true or false?
- If  $\vec{u} \times \vec{v} = \vec{0}$  and  $\vec{u} \cdot \vec{w} = 0$ , then one of  $\vec{v}$  and  $\vec{w}$  is equal to  $\vec{0}$

5. The physics meaning of  $\vec{a} \cdot (\vec{b} \times \vec{c})$ . The volume of the parallelepiped determined by these three vectors.

6. The distance between point and line:

If  $P$  is a point in space and  $\vec{u}$  is a vector, there is a line  $\ell$  which goes through  $P$  and is parallel to  $\vec{u}$ . The line can be described concretely as

the set of all points  $R$  such that  $\vec{PR} = t\vec{u}$  for some  $t \in \mathbb{R}$ . Sometimes you may also see the line written as  $\ell = \{\vec{OP} + t\vec{u} : t \in \mathbb{R}\}$ .

Question: If  $Q$  is another point in space, what the distance between  $Q$  and this line  $\ell$ .

$$\sum_{i=1}^{m_c} (\rho_{-1} \lambda_u)^{2i} \sum_{m=0}^{m_c-i} \Psi(\{u, m\}) \geq \sum_{i=1}^{m_c} (\rho_{-1} \lambda_u)^{2i} \left(1 - \frac{i}{m_c}\right)$$

isn't that

$$\sum_{m=0}^{m_c-i} \Psi(\{u, m\}) \geq 1 - \sum_{m \geq m_c-i} \Psi(\{u, m\}) \geq 1 - c\rho^{3/2}$$

# Note Three , Math21A

J. Yin

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## 1 Equations of Lines

1. Find the equation for the line that passes through points  $P_1 = (1, 2, 3)$  and  $P_2 = (4, 6, 5)$ .
  - (a) Two points determine a line.
  - (b) Point  $Q = (x, y, z)$  is on this line if and only if  $\vec{P_1Q}$  is parallel to  $\vec{P_1P_2}$ , i.e.,

$$\langle x - 1, y - 2, z - 3 \rangle = t \langle 4 - 1, 6 - 2, 5 - 3 \rangle \quad (1)$$

2. Three equivalent Equations.

- (a) Vector Equation
- (b) Parametric Equation
- (c) Symmetric Equation

3. Example: Find a vector Equation for the line that passes through the point  $(5, 1, 3)$  and is parallel to the vector  $\vec{i} + \vec{j} - \vec{k}$

## 2 Equations of Planes

1. How can we describe a plane in space?

A plane in space is determined by a point  $P_0(x_0, y_0, z_0)$  in the plane and a vector  $\vec{n}$  that is orthogonal to the plane.

$$\vec{n} \cdot \vec{P_0P_1} = 0 \Leftrightarrow \text{The point } P_1 \text{ is in the plane}$$

2. Property and Example: Find an equation of the plane that passes through the points  $P(1, 3, 2)$ ,  $Q(3, -1, 6)$  and  $R(5, 2, 0)$

- (a) What is  $\vec{n}$  for this plane?  
(b) In what point does this plane intersect with  $xy$ -plane?
3. The angle between two planes.
- (a) The angle between two planes is defined as the angle between  $\vec{n}_1$  and  $\vec{n}_2$ .  
(b) Find the angle between the planes  $x+y+z = 2$  and  $x-3y+z = 9$ .
4. The Distances
- (a) Find the distance from the point  $(0, 1, 1)$  to the plane  $2x + 3y + 4z = 15$ .  
Draw a line.  
(b) Find the distance from the point  $(1, 3, 2)$  to the line  $x/3 = y - 1 = z + 2$ .  
Draw a plane.

### 3 Functions and Graph

1. Function of two variables:

Q: Find the domain and range of the following function.

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

2. Graph:

If  $f$  is a function of two variables with domain  $D$ , then the graph of  $f$  is the set of all points  $(x, y, z)$  in  $\mathbb{R}^3$  such that  $z = f(x, y)$  and  $(x, y)$  is in  $D$ .

Q: Sketch the graph of the function  $f(x, y) = 6 - 3x - 2y$

# Note Four , Math21A

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## 1 Functions and Graph

1. Function of two variables:

Q: Find the domain and range of the following function.

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

Q: Sketch the graph of this function.

2. Graph:

If  $f$  is a function of two variables with domain  $D$ , then the graph of  $f$  is the set of all points  $(x, y, z)$  in  $\mathbb{R}^3$  such that  $z = f(x, y)$  and  $(x, y)$  is in  $D$ .

Q: Sketch the graph of the function  $f(x, y) = y^2$ .

3. Traces of the surfaces

Keep  $x$  fixed by  $x = k$  and letting  $y$  vary, the result is  $z = \sqrt{4 - k^2 - y^2}$ . This curve is the cross section of the graph and the plane  $x = k$ .

Similarly we can fix  $y$  or  $z$ . All three types of curves are called **Traces**.

Q: Sketch the traces  $z = k$  for  $f(x, y) = \sqrt{4 - x^2 - y^2}$

Q: Match the traces with the graphs.

4. Quadric surface

The set of the points  $(x, y, z)$  which satisfy some fixed second-degree equation in three variable  $x, y, z$ .

Q: What is the Quadric surface for  $z = \sqrt{4 - x^2 - y^2}$ , how about  $z = -\sqrt{4 - x^2 - y^2}$  and  $z^2 + x^2 + y^2 = 4$ .

Q: Sketch the quadric surface with equation

$$x^2 + y^2/9 + z^2/4 = 1 \quad (1)$$

and

$$(x-1)^2 + (y+2)^2/9 + (z-4)^2/4 = 1 \quad (2)$$

5. Some graphs of quadric surfaces pg 682

- (a) Ellipsoid:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- (b) Hyperboloid of one sheet:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- (c) Hyperboloid of two sheet:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$
- (d) Cone  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
- (e) Elliptic Paraboloid  $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- (f) Hyperboloid paraboloid

$$g) \frac{x^2}{a^2} + 2x + cy + fz = 0$$

# Note Five , Math21A

J. Yin

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## 1 Vector Functions

1. Input: real number
2. Output: Vector
3. Simple Example: The parameter equation for a line that passes through  $(1, 2, 1)$  and parallel to  $\vec{n}: (1, 3, 5)$

$$\vec{r}(t) = \langle 1, 2, 1 \rangle + t\langle 1, 3, 5 \rangle = (1+t)\vec{i} + (2+3t)\vec{j} + (1+5t)\vec{k} \quad (1)$$

4. Example 2: Helix.

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k} \quad (2)$$

## 2 Space Curves

In the example of Helix, you are given a vector function and asked for geometric description. Now, you are given a geometric description of a curve and are asked for a parametric equation.

1. Find a vector Function that represents the curve of intersection of the cylinder  $x^2 + y^2 = 1$  and the plane  $y + z = 2$ .

Q:

$$\cos t\vec{i} + \sin t\vec{j} + (2 - \sin t)\vec{k}$$

2. Match the functions and curves.

### 3 Derivatives of Vector Function

1. What is the derivative of Vector Function?

Derivative of normal function

$$\frac{df(t)}{dt} = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \quad (3)$$

Derivative of Vector function

$$\frac{d\vec{r}(t)}{dt} = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \quad (4)$$

2. Limit.
3. Tangent Vector.
4. How can we calculate it?

If  $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ , we have:

$$\vec{r}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} \quad (5)$$

A: Find the parametric equation for the line tangent to the helix we discussed at the point  $(1,0,2\pi)$ .

### 4 Integrals of Vector Function

1. What is the integrals of Vector Function?

$$\int_a^b \vec{r}(t) dt = \int_a^b f(t)\vec{i} dt + \int_a^b g(t)\vec{j} dt + \int_a^b h(t)\vec{k} dt \quad (6)$$

2. Derivative and Integral. Let  $\vec{r}(t)$  be a continuous function. If

$$\int_a^b \vec{r}(t) dt = \vec{R}(b) - \vec{R}(a) \quad (7)$$

then  $\vec{R}'(t) = \vec{r}(t)$ .

3. Position and Speed.

# Note Six , Math21A

J. Yin

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## 1 Arc length

### 1.1 Definition

The limit of the length of inscribed polygons.

### 1.2 Formula

If  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ , then the length from  $\vec{r}(a)$  to  $\vec{r}(b)$  is equal to

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \quad (1)$$

or

$$L = \int_a^b |\vec{r}'(t)| dt \quad (2)$$

Example: Find the length of the arc of the helix from point  $(1, 0, 0)$  to point  $(1, 0, 2\pi)$ .

**Important:** The arc length is independent of the parametrization. We can define a arc length function  $s(t)$  (it is not a vector function)

$$s(t) = \int_a^t |\vec{r}'(u)| du, \quad s'(t) = |\vec{r}'(t)| \quad (3)$$

### 1.3 Parameterize a curve with respect to arc length

Example: Helix again.

$$\vec{r}(s) = \cos(s/\sqrt{2})\vec{i} + \sin(s/\sqrt{2})\vec{j} + s/\sqrt{2}\vec{k} \quad (4)$$

What is the arc length between  $\vec{r}(1.6)$  and  $\vec{r}(2.7)$ .

## 2 Curvature

Unit tangent vector:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad (5)$$

Curvature is defined as

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \quad \text{or} \quad \kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \quad (6)$$

Curvature measures how fast this vector rotates.

### 2.1 Examples

1. A: Calculate the Curvature of straight line.
2. A: Calculate the Curvature of a circle of radius  $a$ .

We can also use the following formula:

$$\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \quad (7)$$

## 3 Normal and Binormal Vectors

First we have  $|\vec{T}(t)| = 1$  and  $\vec{T}'(t) \cdot \vec{T}(t) = 0$ .

### 3.1 Normal vector

Normal vector is defined as

$$\vec{N}(t) \equiv \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \quad (8)$$

So  $\vec{N}(t) \cdot \vec{T}(t) = 0$ .

### 3.2 Binormal vector

$$\vec{B}(t) \equiv \vec{T}(t) \times \vec{N}(t) \quad (9)$$

What is the magnitude of  $\vec{B}(t)$ ?

### 3.3 normal plane and osculating plane

1. Normal plane at  $P = \vec{r}(t)$  is the plane that pass through  $P$  and orthogonal to  $\vec{T}(t)$ . So  $\vec{B}(t)$  and  $\vec{N}(t)$  is parallel to this plane.
2. Osculating plane at  $P = \vec{r}(t)$  is the plane that pass through  $P$  and orthogonal to  $\vec{B}(t)$ . So  $\vec{T}(t)$  and  $\vec{N}(t)$  is parallel to this plane.

A: Calculate the Osculating plane for a circle on  $x - y$  plane.

# Note Seven , Math21A

J. Yin

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## 1 Polar coordinates

Polar Coordinates is a two-dimensional coordinate system.

Point  $(x, y)$  can be represented by  $(r, \theta)$  s.t.

$$x = r \cos \theta \text{ and } y = r \sin \theta \quad (1)$$

We can see that  $r$  and  $\theta$  can be obtained as

$$r^2 = x^2 + y^2 \text{ and } \tan \theta = y/x \quad (2)$$

1.  $r$  is the distance between  $P$  and origin  $O$ .
2.  $\theta$  is the angle between  $\vec{OP}$  and  $x$ -axis.

Q: In polar coordinates, what shapes are described by  $r = k$  and  $\theta = k$ , where  $k$  is a constant?

## 2 Cylindrical coordinates

Cylindrical Coordinates is a three-dimensional coordinate system.

Point  $P: (x, y, z)$  can be represented by  $(r, \theta, z)$  s.t.

$$x = r \cos \theta, \quad y = r \sin \theta \text{ and } z = z \quad (3)$$

If we project Point  $P$  on  $xy$  plane, i.e.  $\vec{PP'}$  is orthogonal to  $xy$  plane and  $P' \in xy$  plane, we can see

1.  $z$  is the distance between  $P$  and  $xy$  plane.
2.  $r$  is the distance between  $P'$  and origin  $O$ .

- $\theta$  is the angle between  $\vec{OP}'$  and  $x$ -axis.

Exercise:

- Q: In cylindrical coordinates, what shapes are described by  $r = k$ ,  $\theta = k$  and  $z = k$  where  $k$  is a constant?
- Q: The cylindrical coordinates are  $(2, 2\pi/3, 1)$ , find its rectangular coordinates.
- Q: The rectangular coordinates are  $(3, 3, 3)$ , find its cylindrical coordinates.
- Q: Find an equation in cylindrical coordinates for the ellipsoid:  $5x^2 + 5y^2 + z^2 = 1$

### 3 Spherical coordinates

Spherical Coordinates is a three-dimensional coordinate system.

Point  $P: (x, y, z)$  can be represented by  $(\rho, \theta, \phi)$ . If we project Point  $P$  on  $xy$  plane, i.e.  $\vec{PP}'$  is orthogonal to  $xy$  plane and  $P' \in xy$  plane, we have

- $\rho$  is the distance between  $P$  and origin  $O$
- $\phi$  is the angle between  $\vec{OP}$  and  $z$ -axis.
- $\theta$  is the angle between  $\vec{OP}'$  and  $x$ -axis.

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta \quad \text{and} \quad z = \rho \cos \phi \quad (4)$$

Exercise:

- Q: In polar coordinates, what shapes are described by  $r = k$ ,  $\theta = k$  and  $z = k$  where  $k$  is a constant?
- Q: The spherical coordinates are  $(2, \pi/4, \pi/3)$ , find its rectangular coordinates.
- Q: The rectangular coordinates are  $(0, 2\sqrt{3}, -2)$ , find its spherical coordinates.
- Q: Find an equation in rectangular coordinates for the surface whose spherical equation is  $\rho = \sin \theta \sin \phi$ .
- Q: Find an inequality for the solid that remains when a hole of radius 2 is drilled through the center of a sphere of radius 5.
- Q: The great-cycle distance between  $P : (\rho, \theta_1, \phi_1)$  and  $Q : (\rho, \theta_2, \phi_2)$ .

## 4 Parametric Surface

Recall the curve:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad (5)$$

Curve is one-dimensional object, so the functions for curves have one variable.

Surface is two-dimensional object, so the functions for surface have two variable.

Parametric Surface:

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad (6)$$

Q: Sketch the surface with vector function:

$$\vec{r}(u, v) = \langle 2 \cos u, v, 2 \sin u \rangle \quad (7)$$

Q: Sketch the surface with vector function:

$$\vec{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle \quad (8)$$

## 5 Grid curves

If we keep  $v$  constant by putting  $v = v_0$ , we get a curve given by  $\vec{r}(u, v_0)$  that lies on the surface  $\vec{r}(u, v)$ . We call these curves grid curves.

# Note Eight, Math21A

J. Yin

April 29, 2009

## 1 Parametric Surface

Recall the curve:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad (1)$$

Curve is one-dimensional object, so the functions for curves have one variable.

Surface is two-dimensional object, so the functions for surface have two variable.

Parametric Surface:

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad (2)$$

Q: Sketch the surface with vector function:

$$\vec{r}(u, v) = \langle 2 \cos u, v, 2 \sin u \rangle \quad (3)$$

Q: Sketch the surface with vector function:

$$\vec{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle \quad (4)$$

### 1.1 Grid curves

If we keep  $v$  constant by putting  $v = v_0$ , we get a curve given by  $\vec{r}(u, v_0)$  that lies on the surface  $\vec{r}(u, v)$ . We call these curves grid curves.

### 1.2 Surface of Revolution

A surface generated by rotating a two-dimensional curve about an axis.

Examples: Sphere, Cone.

## 2 Functions of Several Variables

Q: Find a function which has more than 100 variables.

### 2.1 Contour Maps

1. A contour line (also level set, isopleth, isoline, isogram or isarithm) of a function of two variables is a curve along which the function has a constant value.
2. A contour map is a map illustrated with contour lines.

Examples.

## 3 Limit

### 3.1 Definition

We say the limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  is  $L$  if we can make the values of  $f(x, y)$  as close to  $L$  as we like by taking the point  $(x, y)$  sufficiently close to the point  $(a, b)$ , but not equal to  $(a, b)$ . We say

$$\lim_{x \rightarrow a, y \rightarrow b} f(x, y) = L \quad (5)$$

Example: Show that  $\frac{x^2 - y^2}{x^2 + y^2}$  does not exist.

## 4 Continuity

### 4.1 Definition

A function  $f$  of two variables is called continuous at  $(a, b)$ , if

$$\lim_{x \rightarrow a, y \rightarrow b} f(x, y) = f(a, b) \quad (6)$$

We say  $f$  is continuous on  $D$  if  $f$  is continuous at every point  $(a, b)$  in  $D$ .

### 4.2 Method 1

Polynomial function of two variables is a sum of terms of the form  $cx^m y^n$ , where  $c \in \mathbb{R}$  and  $m, n$  are nonnegative integers. Rational function is a ratio of two polynomials.

1. All polynomials are continuous at every point in  $\mathbb{R}^2$ .
2. Rational function is continuous at the point where the denominator is not equal to zero.

Example: Check the continuity of  $f(x, y) = xy/(x^2 + y^2)$  at  $(0, 0)$ , if  $f(0, 0) = 0$

### 4.3 Method 2

Replace  $x, y$  with  $a + t$  and  $b + ct, c \in \mathbb{R}$ . If for some  $c$ ,

$$\lim_{t \rightarrow 0} f(a + t, b + ct) \neq f(a, b) \text{ or } = \infty$$

we say the  $f$  is not continuous at  $(a, b)$ .

Example: Check the continuity of  $f(x, y) = xy/(x^2 + y^2)$  at  $(0, 0)$ , if  $f(0, 0) = 0$

### 4.4 Method 3

If  $f(x, y) = f(g(x, y))$ ,  $g_0 = g(a, b)$ ,  $g(x, y)$  is continuous at  $(a, b)$  and  $f(g)$  is continuous at  $g_0$ , we say the  $f$  is continuous at  $(a, b)$ .

Example  $f(x, y) = \sin(x^2 + y^2)$ .

# Note Nine, Math21A

J. Yin

April 29, 2009

## 1 Derivative

### 1.1 One variable

Recall the definition of functions  $f(x)$  of one variable,

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad (1)$$

It is the derivative of  $f$  (with respect to  $x$ ) at  $x = a$ .

### 1.2 Two variables

If  $f$  is a function of two variables, we let one variable vary while  $y$  fixed, like  $f(x, b) = g(x)$ , if  $g(x)$  has a derivative at point  $x = a$ , we call it the **partial derivative of  $f$  with respect to  $x$  at  $(a, b)$**  and denote it by  $f_x(a, b)$ . With this definition

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \quad (2)$$

Similarly, **partial derivative of  $f$  with respect to  $y$  at  $(a, b)$**  is defined as

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h} \quad (3)$$

If we now let point  $(a, b)$  vary, we obtain the derivatives in different points. They become functions

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \quad (4)$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} \quad (5)$$

### 1.3 Notations

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_1 = D_1 f = D_x f \quad (6)$$

Examples:

1.  $f(x, y) = x^2 - y^2 + 2xy$ , find  $f_x(2, 1)$  and  $f_x$
2.  $f(x, y) = \sin[x^2 + y^2]$ , find  $f_y$
3.  $f(x, y) = e^{\sin x}(\cos y - 1)$ , find  $f_x(7, 0)$ .

## 2 Derivative of Functions of more than two variable

If  $f$  is a function of three variables,  $f_x$  is defined as

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h} \quad (7)$$

## 3 Higher Derivatives

The derivative  $f_x$  and  $f_y$  are also functions, so we can consider their derivatives  $(f_x)_x$ ,  $(f_y)_x$ ,  $(f_x)_y$  and  $(f_y)_y$ , which are called second partial derivatives of  $f$ . We denote them as  $f_{xx}$ ,  $f_{yx}$ ,  $f_{xy}$  and  $f_{yy}$

Examples:  $f(x, y) = x^3 + y^3 + 4x^2y^2$ , find  $f_{xy}$  and  $f_{yx}$ .

## 4 Partial Differential Equations

Question 1: Find the functions  $u(x, t)$  which satisfy

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad (8)$$

Question 2: Find the functions  $u(x, t)$  which satisfy

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (9)$$

and

$$u(x, 0) = x^2 \quad (10)$$

1. Real Question 1: What is  $u_t(3, 0)$ ?
2. Real Question 2: Check  $u(x, t) = \sin(x - t)$  is one of the solution of the first PDE.

## 5 Linear Approximation

For many function  $f(x, y)$ 's, we have a very useful approximation as follows. For  $(x, y)$  near  $(a, b)$ , we have

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (11)$$

It is called linear approximation at  $(a, b)$ .

Example:  $f(x, y) = xe^{xy}$ , find its linear approximation at  $(1, 0)$  and use it to approximate at  $(1.1, -0.1)$ .

$$f_x = e^{xy} + xye^{xy}, \quad f_y(x, y) = x^2e^{xy} \quad (12)$$

$$f_x(1, 0) = 1, \quad f_y(1, 0) = 1 \quad (13)$$

So we obtain  $f(1.1, -0.1) \approx 1$ , and actual value is 0.98542.

When can we use this linear approximation? First we denote

$$\Delta x = (x - a), \quad \Delta y = (y - b), \quad \Delta z = f(x, y) - f(a, b) \quad (14)$$

We say  $f$  is **differentiable** at  $(a, b)$  if  $\Delta z$  can be written as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \quad (15)$$

where  $\epsilon_1$  and  $\epsilon_2 \rightarrow 0$  when  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

Of course when  $f$  is differentiable, we can use the linear approximation. We have the following easy way to check the differentiability of a function.

**If  $f_x$  and  $f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable.**

# Note Ten, Math21A

J. Yin

April 29, 2009

## 1 Tangent Plane

Recall the definition of curve, tangent vector and tangent line.

$$r(t) = \langle x(t), y(t), z(t) \rangle \quad (1)$$

$$r'(t) = \langle x'(t), y'(t), z'(t) \rangle \quad (2)$$

Consider the surface of  $f(x, y)$ , i.e., the sets  $(x, y, f(x, y))$ . Keep  $x$  fixed, say  $x = a$ , we have a curve on this surface  $(a, y, f(a, y))$ , we can calculate the tangent vector and tangent line at the point  $(a, b)$ . Similarly keep  $y$  fixed, say  $y = b$ , we have another curve  $(x, b, f(x, b))$ , we obtain another tangent line at the point  $(a, b)$ . The tangent plane at point  $(a, b, f(a, b))$  is defined as the plane that passes through  $(a, b)$  and contains these two tangent lines.

We note these two tangent vectors are:

$$\langle 1, 0, f_x(a, b) \rangle \quad (3)$$

$$\langle 0, 1, f_y(a, b) \rangle \quad (4)$$

So the tangent plane at point  $(a, b, f(a, b))$  is perpendicular to

$$\langle 1, 0, f_x(a, b) \rangle \times \langle 0, 1, f_y(a, b) \rangle = \langle -f_x(a, b), -f_y(a, b), 1 \rangle \quad (5)$$

So the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (6)$$

This plane is important because for most function  $f(x, y)$ , locally the points on the plane is very close to the points on the surface.

**For many (not all) functions**, we have the following approximation:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (7)$$

Example:  $f(x, y) = xe^{xy}$ , find its linear approximation at  $(1, 0)$  and use it to approximate at  $(1.1, -0.1)$ .

$$f_x = e^{xy} + xye^{xy}, \quad f_y(x, y) = x^2e^{xy} \quad (8)$$

$$f_x(1, 0) = 1, \quad f_y(1, 0) = 1 \quad (9)$$

So we obtain  $f(1.1, -0.1) \approx 1$ , and actual value is 0.98542.

We can use this approximation when  $f_x$  and  $f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ .

# Note for First Midterm test, Math21A

J. Yin

April 29, 2009

## 1 Vectors and the Geometry of Space

### 1.1 Vectors

### 1.2 Dot Product

Definition(it is number), Dot product of two vectors,

$$\vec{A} \cdot \vec{B} = \cos \theta |\vec{A}| |\vec{B}| \quad (1)$$

$\vec{A} \cdot \vec{B} = 0$  means  $\vec{A}$  is orthogonal to  $\vec{B}$ .  $comp_A B$ ?(number)  $proj_A B$ ?(vector)

### 1.3 Cross Product

Definition(it is vector), Cross product of two vectors,

$$|\vec{A} \times \vec{B}| = \sin \theta |\vec{A}| |\vec{B}| \quad (2)$$

$\vec{A} \times \vec{B}$  is orthogonal to  $\vec{A}$  and  $\vec{B}$ .

### 1.4 Plane and Line

1. Equation for Line: (one point plus direction), (two points)
2. Equation for plane: (one point plus direction), (one point plus two directions) and (three points)
3. Distances between different objections.
4. Closest point.

## 1.5 Functions and Surfaces

What is Surface? Traces. The quadrics in the Table two, Pg682. Match surfaces with equations. (practice a lot)

## 1.6 Cylindrical and Spherical Coordinates

Relation between different Coordinate System (We reviewed it last time). What are  $r$ ,  $\rho$ ,  $\theta$ ,  $\phi$ ? Write a point in different coordinate system, write an equation in different coordinate system, write a condition in different coordinate system ( $r \leq 2$ ) (practice a lot)

# 2 Vector Functions

## 2.1 Vector Functions and Space Curves

1. Vector Function.
2. Find a vector function that represents some given curve.

## 2.2 Derivative and Integral of vector functions

1. Derivatives of vector functions, integrals of vector functions.
2. Tangent vector, unit tangent vector, tangent line.

## 2.3 Arc length and Curvature

1. Arc length:

$$L = \int_a^b |\vec{r}'(t)| dt \quad (3)$$

2. Curvature  $\kappa(t)$ . The curvature of the circle with radius  $a$ . How to calculate  $\kappa(t)$ . Unit tangent vector, normal vector, binormal vector:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}, \quad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \quad (4)$$

Normal plane (orthogonal to  $T(t)$ ). Binormal plane (orthogonal to  $B(t)$ )

## 2.4 Velocity and Acceleration

Check your Hw.

## 2.5 Parametric Surfaces

1. Function of two parameters.
2. Find the surface or equation for some  $\vec{r}(u, v)$ . Find an equation for  $(\cos u, \sin u, v)$  (practice).
3. Find a parameterization of some given surface. Find  $\vec{r}(u, v)$  of hemisphere. (practice).
4. Grid curves. Draw Grid curves and tell the curves  $u = k, v = k$ .

## 3 Partial Derivative

Check your HW. If  $f_x > 0$ , increase  $x$ ,  $f$  is ?. If  $f_{xx} > 0$ , increase  $x$ ,  $f_x$  is ?

# Note Eleven, Math21A

J. Yin

April 7, 2009

## 1 Linear Approximation

### 1.1 Graph

For many (not all) functions, we have the following approximation:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (1)$$

Example:  $f(x, y) = xe^{xy}$ , find its linear approximation at  $(1, 0)$  and use it to approximate at  $(1.1, -0.1)$ .

$$f_x = e^{xy} + xye^{xy}, \quad f_y(x, y) = x^2e^{xy} \quad (2)$$

$$f_x(1, 0) = 1, \quad f_y(1, 0) = 1 \quad (3)$$

So we obtain  $f(1.1, -0.1) \approx 1$ , and actual value is 0.98542.

We can use this approximation when  $f_x$  and  $f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ .

### 1.2 Function of more than two variables

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) \quad (4)$$

### 1.3 Tangent Plane of Parametric Surface

For parametric surface:

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k} \quad (5)$$

the tangent plane at point  $\vec{r}(u_0, v_0)$  is defined as follows. Keep  $u$  constant by putting  $u = u_0$ , then  $\vec{r}(u_0, v)$  becomes a curve (Grid curve), the tangent vector at point  $\vec{r}(u_0, v_0)$  is

$$\vec{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0, v_0)\vec{k} \quad (6)$$

Similarly, Keep  $v$  constant by putting  $v = v_0$ , then  $\vec{r}(u, v_0)$  becomes a curve (Grid curve), the tangent vector at point  $\vec{r}(u_0, v_0)$  is

$$\vec{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0, v_0)\vec{k} \quad (7)$$

The tangent plane is the plane that passes through  $\vec{r}(u_0, v_0)$  and parallels the tangent vector  $\vec{r}_u(u_0, v_0)$  and  $\vec{r}_v(u_0, v_0)$ .

Example:  $\vec{r}(u, v) = (u^2, v^2, uv)$ . We have:

$$\vec{r}_u = (2u, 0, v) \text{ and } \vec{r}_v = (0, 2v, u) \quad (8)$$

So the tangent plane at point  $\vec{r}(1, 1)$  is the plane that passes through  $(1, 1, 1)$  and parallels the tangent vector  $\langle 2, 0, 1 \rangle$  and  $\langle 0, 2, 1 \rangle$ .

## 1.4 General tangent plane

For a surface  $S$ ,  $P$  is on the surface,  $C$  is a curve that passes through point  $P$  and lies on the surface. Then the tangent line of  $C$  at point  $P$  is lying on the tangent plane of  $S$  at  $P$ .

## 2 Chain Rules

Recall the chain rule for the function of one variable.

Example:  $z = \sin(t^2)$ , find  $dz/dt$ .

A: Consider  $x = t^2$ , so  $z = \sin(x)$  and

$$dz/dt = \frac{dy}{dx} \frac{dx}{dt} = \cos(x)2t = 2t \cos(t^2) \quad (9)$$

### 2.1 Case 1

Let's look at cases:  $z = f(x, y)$  and  $x = x(t)$ ,  $y = y(t)$ . For example

$$z = x^2y, \quad x = \sin t, \quad y = e^t \quad (10)$$

We can see that  $z$  is a function with the variable  $t$ . We have the formula:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad (11)$$

Q: Calculate  $\frac{dz}{dt}$  in (10).

A: We have

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= 2xy \cos t + x^2 e^t \\ &= 2 \sin t e^t + \sin^2 t e^t \end{aligned} \quad (12)$$

## 2.2 Case 2

Let's look at cases:  $z = f(x, y)$  and  $x = x(u, v)$ ,  $y = y(u, v)$ . For example

$$z = x^2 - y^2, \quad x = u + v, \quad y = u - v \quad (13)$$

We can see that  $z$  is a function with the variables  $u, v$ . We have the formula:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad (14)$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \quad (15)$$

Q: Calculate  $\frac{\partial z}{\partial u}$  in (13) with chain rule.

A: We have

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ &= 2x - 2y \\ &= 4v \end{aligned} \quad (16)$$

Check it.

## 2.3 General Version

Suppose that  $u$  is a function of the  $n$  variables  $x_1, x_2, \dots, x_n$  and each  $x_j$  is function of the  $m$  variables  $t_1, t_2, \dots, t_m$ . Then  $u$  is a function of  $t_1, t_2, \dots, t_m$  and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i} \quad (17)$$

Suppose  $u = x^2 + y^2 + z^2$ ,  $x = s^2$ ,  $y = \sin s$  and  $z = e^s$ . Find  $du/ds$

### 3 Application: Implicit Differentiation

Question:  $x^3 + y^3 + xy = 0$ , find  $dy/dx$ . We know it is not easy to obtain the solution  $y = y(x)$ . But we can let  $F(x, y) = x^3 + y^3 + xy$  and  $x = x$ ,  $y = y(x)$ . Using chain rule, we obtain

$$dF/dx = \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} \quad (18)$$

Because  $F(x, y(x)) = 0$ , so  $dF/dx = 0$ . We can see:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \quad (19)$$

Calculate the above example:

$$\frac{dy}{dx} = -\frac{3x^2 - y}{3y^2 - x} \quad (20)$$

Question:  $x^3 + y^3 + z^3 + xyz = 0$ , find  $\partial z/\partial x$ . Let  $F(x, y, z) = x^3 + y^3 + z^3 + xyz$ . We skip the proof, the formula is

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad (21)$$

Example: For  $x + z = yz$ , find  $\frac{\partial z}{\partial x}$ . Choose  $F(x, y, z) = x + z - yz$ .

# Note Twelve, Math21A

J. Yin

April 7, 2009

## 1 Directional Derivatives

The partial derivatives  $f_x$  and  $f_y$  at point  $(x_0, y_0)$  are defined as

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \quad (1)$$

$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \quad (2)$$

These numbers tell the rate of change of  $f$  in the  $x$ - and  $y$ -directions. Now we want to find the rate of change of  $z$  at point  $(x_0, y_0)$  in the some arbitrary direction, like the direction which parrallel to  $\langle a, b \rangle$ .

### 1.1 Definition

The **Directional Derivatives** of  $f$  at  $(x_0, y_0)$  in the direction of a **Unit** vector  $\vec{u} = (a, b)$  is

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} \quad (3)$$

Example:  $f(x, y) = x^2 - y^2$ ,  $\vec{u} = (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$ , Find  $D_{\vec{u}}f(2, 0)$

Theorem: If  $f_x$  and  $f_y$  are continuous function, we have

$$D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \quad (4)$$

and

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b \quad (5)$$

Check the example we calculated.

## 1.2 Gradient Vector

We can see the equality (5) can be written as

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle, \quad (6)$$

which means if we know  $\langle f_x, f_y \rangle$ , we can calculate the rate of change of  $f$  in different directions easily. We define **Gradient Vector** as this vector  $\langle f_x, f_y \rangle$ , i.e.,

$$\nabla f = \langle f_x, f_y \rangle \quad (7)$$

It is read *del f*.

## 1.3 Function of Three variables

For unit vector  $\vec{u}$ (three dimensional),

$$D_{\vec{u}}f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h} \quad (8)$$

and

$$D_{\vec{u}}f(x_0, y_0, z_0) = \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle \quad (9)$$

Example:  $f(x, y, z) = x^2 + y^2 + z^2$ , find the directional derivative of  $f$  at  $(1, 3, 0)$  in the direction of  $\langle 2, 0, 1 \rangle$ .

## 1.4 Maximizing the Directional Derivative

The maximum value of the directional derivative  $D_{\vec{u}}f(x_0, y_0, z_0)$  is equal to  $|\nabla f(x_0, y_0, z_0)|$ . It occurs when  $\vec{u}$  has the same direction as the gradient vector  $\nabla f(x_0, y_0, z_0)$ .

Example:  $T(x, y) = x^2 - y$ , you are at  $(1, 3)$ , you are moving toward  $(5, 4)$ , it is getting colder or warmer. In which direction, you will feel warmer fastest?

## 2 Tangent Line to level curve

For  $g(x, y) = x^2 + y^2$ , draw the level curve(contour curve)

$$g(x, y) = x^2 + y^2 = 2.$$

Find the tangent vector and gradient vector at  $P = (1, 1)$ . For this curve, we choose  $\vec{r}(t) = \sqrt{2} \cos t \vec{i} + \sqrt{2} \sin t \vec{j}$ . We obtain :

$$\vec{r}'(\pi/4) = \sqrt{2} \vec{i} - \sqrt{2} \vec{j}, \quad \nabla g(1, 1) = 2\vec{i} + 2\vec{j} \quad (10)$$

We have:

$$r' \cdot \nabla g = 0 \quad (11)$$

Example (When you know  $\nabla g$ , in the plane, you can find the tangent line.)

### 3 Tangent Planes to level surfaces

For  $g(x, y, z) = x^2 + y^2 + z^2$ , draw the level surface

$$g(x, y, z) = x^2 + y^2 + z^2 = 3.$$

Find the tangent plane and gradient vector at point  $P = (1, 1, 1)$ . We find the gradient vector is perpendicular to the tangent plane at the same point.

**Normal line** to Surface  $S$  at  $P$  is the line passing through  $P$  and perpendicular to the tangent plane.

Examples.

# Note Thirteen, Math21A

J. Yin

April 29, 2009

## 1 Maximum and Minimum Values

It is easy to understand global(absolute) Maximum and Minimum. Let's talk about Local Maximum and Minimum.

### 1.1 Local Maximum and Minimum

A function of two variables has a **local maximum** at  $(a, b)$  if there exists a disk  $D$  with the center  $(a, b)$ . For any other  $(x, y)$  in this disk, we have:

$$f(x, y) \leq f(a, b), \quad (x, y) \in D \quad (1)$$

And the number  $f(a, b)$  is called a **local maximum value**.

Similarly, we can define **local minimum** and **local minimum value**.

Question: Is global maximum local maximum?

Answer: True.

### 1.2 Find the Maximum and Minimum

Recall the method we used for finding the Maximum and Minimum for the function  $f(x)$  of one variable. If  $f$  has a local maximum or minimum at  $x = a$ , then

$$f'(a) = 0 \quad (2)$$

**THEOREM 1.** *If  $f$  has a local maximum or minimum at  $(a, b)$  and the first-order partial derivatives of  $f$  exist there. Then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ , i.e.*

$$\nabla f(a, b) = 0 \quad (3)$$

*Proof.* If  $f$  has a local maximum or minimum at  $(a, b)$ , we fix  $y = b$ . Then the function of one variable  $g(x) = f(x, b)$  has local maximum or minimum at  $x = a$ . With Fermat's theorem, we obtain  $g'(a) = 0$ , i.e.  $f_x(a, b) = 0$ .  $\square$

Note:  $\nabla f(a, b) = 0$  does not implies that  $f$  has a local maximum or minimum at point  $(a, b)$

The point  $(a, b)$  is called **critical point**(stationary point) when

$$\nabla f(a, b) = 0.$$

Find the critical point of function  $f(x, y) = y^2 - x^2$ . It is  $(0, 0)$ , but it is neither local Maximum or Minimum.

A **saddle point** is a critical point but not local Maximum or Minimum.

### **THEOREM 2.** *Second Derivative Test*

*Suppose the second partial derivative of  $f$  are continuous on a disk with center  $(a, b)$ , and suppose  $\nabla f(a, b) = 0$ , i.e.  $(a, b)$  is a critical point of  $f$ . Let:*

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \quad (4)$$

*We have:*

- 1. If  $D > 0$  and  $f_{xx}(a, b) > 0$ ,  $f(a, b)$  is a local minimum.*
- 2. If  $D > 0$  and  $f_{xx}(a, b) < 0$ ,  $f(a, b)$  is a local maximum.*
- 3. If  $D < 0$ , then  $f(a, b)$  is a saddle point.*
- 4. If  $D = 0$ , maybe this, maybe that.*

Examples:

1. Find all critical points of  $f(x, y) = xy^2 - x^2 - 2y^2$  and determine whether each is a local minimum , local maximum, or saddle point.  
Answer: Three critical points  $(0, 0)$ (local maximum) and  $(2, \pm 2)$  (saddle point).
2. Find all critical points with the contour map and determine whether each is a local minimum , local maximum, or saddle point.

## **2 Global (Absolute) Maximum or Minimum**

**A closed set** in  $\mathbb{R}^2$  is one that contains all its boundary points. For example: What is the boundary of unit disk?

**A bounded set** in  $\mathbb{R}^2$  is one that can be covered by some disk. For example: is  $x$ -axis bounded?

**THEOREM 3.** *If  $f$  is continuous on a closed and bounded set  $D$ . Then  $f$  attains an absolute maximum value and an absolute minimum value*

To find the absolute maximum value or absolute minimum value, we only need to

1. Find all the critical points in  $D$ .
2. Find the extreme points on Boundary.
3. Compare the points you obtain in step 1 and step 2.

Example: Find the absolute maximum and minimum values of  $f(x, y) = y^2 - x^2$  on square  $|x| \leq 1, |y| \leq 1$ .

Example: Find the absolute maximum and minimum values of  $f(x, y) = x + y$  on disk  $x^2 + y^2 \leq 1$ .

### 3 Lagrange Multiple

Question: Find the maximum and minimum volumes of a rectangular box whose surface area is  $1500\text{cm}^2$ .

Maximize  $V = xyz$  subject to the constraint  $2xy + 2yz + 2xz = 1500$ .

Question: To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$ .

1. Find all values of  $x, y, z$  and  $\lambda$ (something new!) such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = k \quad (5)$$

Note: Four variables and four equations, so we can find solutions.

2. Evaluate  $f$  at all the points  $(x, y, z)$  that result from last step. The largest one is maximum and smallest one is minimum.

Example : Maximize  $xyz$  subject to the constraint  $x + y + z = 1$ .

We let  $f = xyz$  and  $g = x + y + z$ . Then we have:

$$\nabla f = \langle yz, xz, zy \rangle \quad (6)$$

and

$$\nabla g = \langle 1, 1, 1 \rangle \quad (7)$$

Solve (5), we obtain that

$$x = y = z = 1/3 \text{ and } \lambda = 1/9 \quad (8)$$

# Note Fourteen, Math21A

J. Yin

April 7, 2009

## 1 Review

1. The point  $(a, b)$  is called **critical point**(stationary point) when

$$\nabla f(a, b) = 0.$$

2. A critical point is a **local Maximum** or **local Minimum** or **saddle point**(concave up in one direction and concave down in another).

3. **Second Derivative Test:**

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \quad (1)$$

We have:

- (a) If  $D > 0$  and  $f_{xx}(a, b) > 0$ ,  $f(a, b)$  is a local minimum.
  - (b) If  $D > 0$  and  $f_{xx}(a, b) < 0$ ,  $f(a, b)$  is a local maximum.
  - (c) If  $D < 0$ , then  $f(a, b)$  is a saddle point.
  - (d) If  $D = 0$ , maybe this, maybe that.
4. Find the critical points on contour map.
    - (a) Circle: Local minimum or maximum
    - (b) Cross: Saddle point.
  5. **Global (Absolute) Maximum or Minimum.** To find the absolute maximum value or absolute minimum value, we only need to
    - (a) Find all the critical points in  $D$ .
    - (b) Find the extreme points on Boundary.
    - (c) Compare the points you obtain in step 1 and step 2.

## 2 Lagrange Multipliers

Question: The distance function:

$$f(x, y, z) = \sqrt{(x-1)^2 + (y-1)^2 + (z-3)^2} \quad (2)$$

1. What is the minimum value of this function?
2. If we restrict the points on the sphere  $x^2 + y^2 + z^2 = 1$ , what is the minimum value?
3. If on ellipsoid  $x^2 + 2y^2 + 3z^2 = 1$ ?

Question: To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$ .

1. Find all values of  $x, y, z$  and  $\lambda$ (something new!) such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = k \quad (3)$$

Note: Four variables and four equations, so we can find solutions.

2. Evaluate  $f$  at all the points  $(x, y, z)$  that result from last step. The largest one is maximum and smallest one is minimum.

Example : Maximize  $xyz$  subject to the constraint  $x^2 + y^2 + z^2 = 1$ .

We let  $f = xyz$  and  $g = x + y + z$ . Then we have:

$$\nabla f = \langle yz, xz, zy \rangle \quad (4)$$

and

$$\nabla g = \langle 2x, 2y, 2z \rangle \quad (5)$$

Solve (3).

**Note:** For the functions  $f, g$  of two variables, we still only need to solve:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = k. \quad (6)$$

If  $x_0, y_0$  and  $\lambda_0$  is the solution, then on the contour map of  $f$  and  $g$  are tangent to each other at point  $x_0, y_0$ , or we say the contour line  $f(x, y) = f(x_0, y_0)$  and  $g(x, y) = k$  are tangent to each other at point  $x_0, y_0$ .

Example:  $f = 4x^2 + y, g(x, y) = x^2 + y^2/4$ . The constraint is  $g(x, y) = 1$ ?

We need to solve

$$8x = \lambda 2x \quad (7)$$

$$1 = \lambda y/2$$

$$x^2 + y^2/4 = 1$$

The solution is  $x_0 = \pm\sqrt{15}/4, y_0 = 1/2$ , that is it ??

### 3 Two Constraints

Question: To find the maximum and minimum values of  $f(x, y, z)$  subject to two constraints:  $g(x, y, z) = k$  and  $h(x, y, z) = c$ .

1. Find all values of  $x, y, z$  and  $\lambda$  and  $\mu$  (something new!) such that

$$\begin{aligned}\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) &= k \text{ and } h(x, y, z) = c\end{aligned}\quad (8)$$

In other words, find all values of  $x, y, z$  and  $\lambda, \mu$  such that

$$\begin{aligned}f_x(x, y, z) &= \lambda g_x + \mu h_x \\ f_y(x, y, z) &= \lambda g_y + \mu h_y \\ f_z(x, y, z) &= \lambda g_z + \mu h_z \\ g(x, y, z) &= k \\ h(x, y, z) &= c\end{aligned}\quad (9)$$

Five variables and Five equations, so we can find solutions.

2. Evaluate  $f$  at all the points  $(x, y, z)$  that result from last step. The largest one is maximum and smallest one is minimum.

Example: Find the maximum value of the function  $f(x, y, z) = x + 2y + 3z$  on the curve of intersection of the plane  $x - y + z = 1$  and the cylinder  $x^2 + y^2 = 1$ .

We need to solve:

$$\begin{aligned}1 &= \lambda + 2x\mu \\ 2 &= -\lambda + 2y\mu \\ 3 &= \lambda \\ x - y + z &= 1 \\ x^2 + y^2 &= 1\end{aligned}\quad (10)$$

The solution is  $x = \mp 2/\sqrt{29}$ ,  $y = \pm 5/\sqrt{29}$  and  $z = 1 \pm 7/\sqrt{29}$ . The maximum is  $3 + \sqrt{29}$  and the minimum is  $3 - \sqrt{29}$ .

### 4 HW

Hint for question 40(b).

# Note Fifteen, Math21A

## Multiple Integrals

J. Yin

April 29, 2009

### 1 Definition

#### 1.1 Review of the Definite Integral

First let's recall the definite integral of the function of one variable. If  $f(x)$  is defined for  $a \leq x \leq b$ . We start by dividing the interval  $[a, b]$  into  $n$  subintervals  $[x_{i-1}, x_i]$  of equal width  $\Delta x = [b - a]/n$  and choose sample points  $x_i^*$  in these subintervals. Then we obtain a number (Riemann sum):

$$\sum_{i=1}^n f(x_i^*) \Delta x \quad (1)$$

take the limit of such sums as  $n \rightarrow \infty$  to obtain the definite integral (Riemann integral) of  $f$  from  $a$  to  $b$ :

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \quad (2)$$

$\Delta x = (b - a)/n$ .

Visit: <http://www.hostsrv.com/webmaa/app1/MSP/webm1010/riemann>.

Note: The area of the  $i$ th rectangle is equal to  $f(x_i^*) \Delta x$ . And the Riemann sum is equal to the total area of the rectangles.

If  $f(x) \geq 0$ , the Riemann integral is the area below  $f(x)$  and  $x$ -axis.

Question: If  $f(x) \not\geq 0$ , how do you explain the Riemann integral.

#### 1.2 Volume and Double Integrals

We consider a function  $f$  of **two variables** defined on a closed rectangle:

$$R = [a, b] \times [c, d] \quad (3)$$

and we first suppose that  $f(x, y) \geq 0$ . The first step is to divide the rectangle  $R$  into small subrectangles. Let's cut  $R$  into  $m \times n$  rectangles, each with area  $\Delta A = \Delta x \Delta y$ . What are  $\Delta x$  and  $\Delta y$ ? Then choose one sample point  $P_{i,j} = (x_{i,j}^*, y_{i,j}^*)$ . Double Riemann sum is:

$$\sum_{i=1}^m \sum_{j=1}^n f(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (4)$$

Double Riemann integral is the limit when you take  $m, n \rightarrow \infty$ , i.e.,

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (5)$$

It can be explained as the volume of solid  $S$  that lies under the graph  $f$  and above the rectangle  $R$ .

Question: Estimate the volume of the solid that lies above the square  $R = [0, 2] \times [0, 2]$  and below the elliptic paraboloid:

$$z = 16 - x^2 - 2y^2.$$

Divide  $R$  into four squares and choose the sample point to be the upper right corner of each square  $R_{i,j}$ .

$$V \approx 13 + 7 + 10 + 4 = 34. \quad (6)$$

Make sure one point for one box.

### 1.3 The Midpoint Rule

As you can guess, The Midpoint Rule means that we choose the midpoint as the sample point of each subrectangle.

Question: Using midpoint rule to estimate the volume of the solid in last question.

Answer:?

### 1.4 Average value

For functions of one variable:

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx \quad (7)$$

For functions of two variables:

$$f_{ave} = \frac{1}{A(R)} \int \int_R f(x, y) dA \quad (8)$$

Of course, we can see:

$$f_{ave} A(R) = \int \int_R f(x, y) dA \quad (9)$$

### 1.5 Some properties

1.

$$\int \int_R f(x, y) + g(x, y) dA = \int \int_R f(x, y) dA + \int \int_R g(x, y) dA \quad (10)$$

2. For constant  $c$ ,

$$\int \int_R cf(x, y) dA = c \int \int_R f(x, y) dA \quad (11)$$

3. If  $f(x, y) \leq g(x, y)$  for all  $(x, y)$  in  $R$ , then

$$\int \int_R f(x, y) dA \leq \int \int_R g(x, y) dA \quad (12)$$

## 2 Iterated Integrals

Now, we are going to study how to calculate the integral on  $R$ .

Suppose that  $f(x, y)$  is a function of two variables that is continuous on the rectangle  $R = [a, b] \times [c, d]$ . We use the notation

$$A(x) = \int_c^d f(x, y) dy \quad (13)$$

to mean that  $x$  is held fixed and  $f(x, y)$  is integrated with respect to  $y$  from  $c$  to  $d$ . (it is one-dimensional integral). Then we integrate  $A(x)$  with respect to  $x$  from  $a$  to  $b$ .

$$\int_a^b A(x) dx = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx \quad (14)$$

The right side is usually written as

$$\int_a^b \int_c^d f(x, y) dy dx \quad (15)$$

Similarly, we can define:

$$\int_c^d \int_a^b f(x, y) dx dy \quad (16)$$

Give you a simple  $f(x, y)$ , you can calculate (15) and (16), right?

The following theorem (Fubini's Theorem) is **very important and useful**,

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy \quad (17)$$

Examples

### 3 Double Integral over General Regions

#### 3.1 Definition

If  $D$  is not a rectangle, first, we find a rectangle  $R$ , which cover  $D$ , we define  $F(x, y)$  on  $R$  as

$$F(x, y) = f(x, y), \text{ when } (x, y) \in D \quad (18)$$

and  $F(x, y) = 0$  when  $(x, y) \notin D$ . Then

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA \quad (19)$$

When  $f \geq 0$ , it still can be explained as the volume. Think about a hemisphere.

Questions.

#### 3.2 Methods we can use

Questions, Calculate the following integral :

$$\int \int_D y dA \quad (20)$$

Here  $D$  is a triangle.

You can see that when  $D$  is said to be **type I**, i.e.,

$$D \equiv \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (21)$$

we can simplify the integral, i.e.,

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (22)$$

Note Figure 5 in pg844.

Example: Evaluate  $\int \int_D x + 2y dA$ , where  $D$  is the region bounded by the parabolas  $y = 2x^2$  and  $y = 1 + x^2$ .

Similarly, by changing  $x$  and  $y$ , we have the following result.

When  $D$  is said to be **type II**, i.e.,

$$D \equiv \{(x, y) | c \leq y \leq d, h_1(x) \leq x \leq h_2(x)\} \quad (23)$$

we can simplify the integral, i.e.,

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y) dx dy \quad (24)$$

# Note Sixteen, Math21A

## Multiple Integrals

J. Yin

April 29, 2009

### 0.1 Last Class

1. Definition of Double Integral
2. Fubini's Theorem:

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy \quad (1)$$

## 1 Double Integral over General Regions

### 1.1 Definition

If  $D$  is not a rectangle, first, we find a rectangle  $R$ , which cover  $D$ , we define  $F(x, y)$  on  $R$  as

$$F(x, y) = f(x, y), \text{ when } (x, y) \in D \quad (2)$$

and  $F(x, y) = 0$  when  $(x, y) \notin D$ . Then

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA \quad (3)$$

When  $f \geq 0$ , it still can be explained as the volume. Think about a hemisphere.  $z = f(x, y) = \sqrt{1 - x^2 - y^2}$  and  $D$  is  $x^2 + y^2 \leq 1$ .

$$\int \int_D \sqrt{1 - x^2 - y^2} dA = \frac{2}{3}\pi \quad (4)$$

Questions.

## 1.2 Methods we can use

Questions, Calculate the following integral :

$$\int \int_D y dA \quad (5)$$

Here  $D$  is a triangle.

You can see that when  $D$  is **type I**, i.e.,

$$D \equiv \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (6)$$

we can simplify the integral, i.e.,

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (7)$$

Note Figure 5 in pg844.

Example: Evaluate  $\int \int_D x + 2y dA$ , where  $D$  is the region bounded by the parabolas  $y = 2x^2$  and  $y = 1 + x^2$ .

Similarly, by changing  $x$  and  $y$ , we have the following result.

When  $D$  is said to be **type II**, i.e.,

$$D \equiv \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\} \quad (8)$$

we can simplify the integral, i.e.,

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy \quad (9)$$

So far, we only see the question whose  $f$  and  $D$  are clearly defined. Let's see something else.

Question: Find the volume of the tetrahedron bounded by the planes  $x + 2y + z = 2$ ,  $x = 2y$ ,  $x = 0$  and  $z = 0$

$$D = \{(x, y) | 0 \leq x \leq 1, x/2 \leq y \leq 1 - x/2\} \quad (10)$$

$$f = 2 - x - 2y \quad (11)$$

$$V = \int \int_D (2 - x - 2y) dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) dy dx = \frac{1}{3} \quad (12)$$

A hint for question 30.

we have:

$$\int \int_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta \quad (20)$$

Let's calculate the volume of the hemisphere with radius 1.

$$\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r dr d\theta = 3\pi/2 \quad (21)$$

Find the volume of the solid bounded by the plane  $z = 0$  and the paraboloid  $z = 1 - x^2 - y^2$

$$\int_0^{2\pi} \int_0^1 (1-r^2) r dr d\theta = \pi/2 \quad (22)$$

### 3.1 general case

If the domain has the form :

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), \} \quad (23)$$

we have:

$$\int \int_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (24)$$

Question: Sketch the  $D$ :

$$D = \{(r, \theta) | 0 \leq \theta \leq \pi/2, 0 \leq r \leq \sin \theta, \} \quad (25)$$

Question: Find the integral for the volume of the solid that lies under the paraboloid  $z = x^2 + y^2$ .

We have  $f(r, \theta) = r^2$ . What is  $D$ ?

$$(x-1)^2 + y^2 \leq 1 \Rightarrow 0 \leq r \leq 2 \cos \theta, \quad -\pi/2 \leq \theta \leq \pi/2 \quad (26)$$

So,

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 r dr d\theta \quad (27)$$

Question: Calculate the area bounded by curve of  $r = \sin(10\theta)$

$$A = 20 \int \int_D 1 dA = 20 \int_0^{\pi/10} \int_0^{\sin(10\theta)} r dr d\theta = \pi/2 \quad (28)$$

If you replace 10 with any other even number, you will see that the area is also equal to  $\pi/2$

# Note for Second Midterm test, Math21A

## From section 11.4 to 12.4

J. Yin

April 7, 2009

Jun got a disease called Bell's palsy, which may last few weeks and get worse. It is definitely not a good news for a teacher.

### 1 Partial Derivatives

For a simple function  $f(x, y)$  or  $f(x, y, z)$ , e.g.  $f(x, y) = \sin(x + y)y^2$ , you can calculate

$$f_x, f_y, f_{x,x}, f_{xy}, f_{yx}, \dots \quad (1)$$

**This is the first step for many questions, so check your calculation.**

We note:  $f_{xy} = f_{yx}$  is not correct for all function  $f(x, y)$ . But when  $f_{xy}, f_{yx}$  are continuous, this result holds.

#### 1.1 Linear approximation

For a function  $f(x, y)$ , you can write the **linear approximation** at the point  $(a, b)$ :

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (2)$$

Then you can use it to estimate the value of  $f(x, y)$  by insert  $x, y$  into the rhs.

#### 1.2 Directional derivatives

**Directional derivative** is the rate of change of  $f(x, y)$  at point  $(x_0, y_0)$  in the some arbitrary direction

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} \quad (3)$$

Note:  $\vec{u}$  is a unit vector. What is unit vector in the direction  $\langle 3, 4 \rangle$ ? Give you a contour map, one point, one direction, can you tell the sign of  $D_{\vec{u}}f$ ? Don't forget

$$f_x = D_{\vec{u}=(1,0)}f, \quad f_y = D_{\vec{u}=(0,1)}f, \quad (4)$$

Do you remember how to calculate  $D_{\vec{u}}f$ ? We need to calculate **Gradient Vector**.

$$\nabla f = \langle f_x, f_y \rangle \quad (5)$$

It is read *del f*. We have if  $\vec{u} = (a, b)$ , again,  $\vec{u}$  is a unit vector, then

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle \quad (6)$$

Note:  $\nabla f$  is a vector (function) and  $D_{\vec{u}}f$  is a number.

Typical question: In fixed point, in which direction does  $z = f(x, y)$  increase fastest, decrease fastest or does not change?

With gradient vector, we can easily obtain the **Tangent line in contour map**, for function  $f(x, y)$  and point  $(a, b)$ , we know the tangent vector  $r$  is orthogonal to  $\nabla f(a, b)$

$$\vec{r} \cdot \nabla f(a, b) = 0 \quad (7)$$

And it is easy for us to obtain **Tangent plane for surface**, for surface  $f(x, y, z) = k$ , e.g.  $x^2 + y^2 + z^2 = 3$ , and point  $(a, b, c) = (1, -1, 1)$ , we know the normal vector of this tangent plane is orthogonal to  $\nabla f(a, b, c)$

$$\vec{r} \cdot \nabla f(a, b, c) = 0 \quad (8)$$

### 1.3 Minimum and Maximum

1. Critical points,  $\nabla f = 0$ . local minimum, local maximum, saddle point. Second derivative test.

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \quad (9)$$

We have:

- (a) If  $D > 0$  and  $f_{xx}(a, b) > 0$ ,  $f(a, b)$  is a local minimum.
  - (b) If  $D > 0$  and  $f_{xx}(a, b) < 0$ ,  $f(a, b)$  is a local maximum.
  - (c) If  $D < 0$ , then  $f(a, b)$  is a saddle point.
  - (d) If  $D = 0$ , maybe this, maybe that.
2. Point out the Critical points on contour map. Tell they are local minimum, local maximum or saddle point. Circle and Cross.

3. Find the maximum or minimum in some region  $D$ .

- (a) Find all the critical points in  $D$ .
- (b) Find the extreme points on Boundary.
- (c) Compare the points you obtain in step 1 and step 2.

### 1.4 Chain rules and implicit Question

Will not be harder than Hws.

### 1.5 Lagrange Multipliers

The typical question: Find the (local) the maximum and minimum values (points) of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$ . (maybe just  $(x, y)$ )

1. Find all values of  $x, y, z$  and  $\lambda$ (something new!) such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = k \quad (10)$$

Note: Four variables and four equations, so we can find solutions.

2. Evaluate  $f$  at all the points  $(x, y, z)$  that result from last step. The largest one is maximum and smallest one is minimum.

What is the meaning of highest? closest?.

Tricks:

1.  $2x = \lambda x$  implies  $\lambda = 2$  or  $x = 0$

2.

$$A = B, C = D \Rightarrow A - C = B - D \quad (11)$$

$$A = B, C = D \Rightarrow A/C = B/D \quad (12)$$

For the functions  $f, g$  of two variables, we still only need to solve:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = k. \quad (13)$$

If  $x_0, y_0$  and  $\lambda_0$  is the solution, then on the contour map of  $f, g = k$  is tangent to  $f$  at point  $x_0, y_0$ , or we say the contour line  $f(x, y) = f(x_0, y_0)$  and  $g(x, y) = k$  are tangent to each other at point  $x_0, y_0$ .

Example:  $f = x^2 + y^2, g(x, y) = y - x^2$ . The constraint is  $g(x, y) = 1$ . One contour line of  $f$  is tangent to  $g = 1$  at point  $(0, 1)$ . This is minimum point. We have the minimum value of  $f = x^2 + y^2$  subject to the constraint  $g(x, y) = y - x^2 = 1$  is one. Where is the maximum point?

## 2 Double Integral

There is a typical step in many questions, i.e., calculating

$$\int_{g_1(x)}^{g_2(x)} f(x, y) dy \quad (14)$$

or  $x \leftrightarrow y$ . Do it carefully.

### 2.1 Definition

### 2.2 Fubini's Theorem

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy \quad (15)$$

**Rieman integral** can be calculated with **iterated integrals**.

If the domain is not a square, if  $D$  is **type I** (3 kinds), i.e.,

$$D \equiv \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (16)$$

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (17)$$

If  $D$  is **type II**?

The question could be give you a domain and function and ask you to calculate it.

We have seen that in some cases you need to change the order of  $dx dy$ , e.g.,

$$\int_0^1 \int_x^1 \sin(y^2) dy dx \quad (18)$$

Some properties:

1. If  $f \geq g$  in  $D$ , then

$$\iint_D f(x, y) dA \geq \iint_D g(x, y) dA \quad (19)$$

- 2.

$$\int_D f(x, y) + g(x, y) dA = \int_D f(x, y) dA + \int_D g(x, y) dA \quad (20)$$

3. If  $D = D_1 \cup D_2$  where  $D_1$  and  $D_2$  don't overlap except perhaps on their boundaries then

$$\iint_D f(x, y) dA = \iint_{D_2} f(x, y) dA + \iint_{D_1} f(x, y) dA \quad (21)$$

Example: Triangle.

### 2.3 Double Integrals in Polar Coordinates

It is new, but it is Important. Check your HWs. For general case, we have:

$$\int \int_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (22)$$

Example, the area bounded by  $r = \theta$  and positive  $x$ -axis, we have  $f(x, y) = 1$  and

$$\int \int_R f(x, y) dA = \int_0^{2\pi} \int_0^{\theta} r dr d\theta \quad (23)$$

## 3 Applications of Double integrals

### 3.1 Density and Mass

One planar lamina has the shape  $D$ , the density is not uniform. The local density is defined as

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A} \quad (24)$$

So the total mass is about:

$$m \approx \sum_{i=1}^m \sum_{j=1}^n \rho(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (25)$$

In the limit  $m, n \rightarrow \infty$ , we have

$$m = \int \int_D \rho(x, y) dA \quad (26)$$

### 3.2 Moments and Centers of Mass

The moment of a particle about an axis as the product of its mass and its distance from the axis. Let's calculate the moment of the lamina. The moment of the entire lamina about the  $x$ -axis is

$$M_x \approx \sum_{i=1}^m \sum_{j=1}^n y_{i,j}^* \rho(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (27)$$

Similarly, the moment about  $y$ -axis:

$$M_y \approx \sum_{i=1}^m \sum_{j=1}^n x_{i,j}^* \rho(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (28)$$

In the limit  $m, n \rightarrow \infty$ , we have,

$$M_x = \int \int_D y \rho(x, y) dA \quad (29)$$

and

$$M_y = \int \int_D x \rho(x, y) dA \quad (30)$$

We define the center of the mass  $(\bar{x}, \bar{y})$ , which is a point,

$$\bar{x} = M_y/m, \quad \bar{y} = M_x/m \quad (31)$$

# Note 18, Math21A

J. Yin

April 30, 2009

Jun got a disease called Bell's palsy, which may last few weeks and get worse. It is definitely not a good news for a teacher.

## 1 Applications of Double integrals

### 1.1 Density and Mass

One planar lamina has the shape  $D$ , the density is not uniform. The local density is defined as

$$\rho(x, y) = \lim_{\Delta A \rightarrow 0} \frac{\Delta m}{\Delta A} \quad (1)$$

So the total mass is about:

$$m \approx \sum_{i=1}^m \sum_{j=1}^n \rho(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (2)$$

In the limit  $m, n \rightarrow \infty$ , we have

$$m = \iint_D \rho(x, y) dA \quad (3)$$

### 1.2 Moments and Centers of Mass

The moment of a particle about an axis is the product of its mass and its distance from the axis. Let's calculate the moment of the lamina. The moment of the entire lamina about the  $x$ -axis is

$$M_x \approx \sum_{i=1}^m \sum_{j=1}^n y_{i,j}^* \rho(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (4)$$

Similarly, the moment about  $y$ -axis:

$$M_x \approx \sum_{i=1}^m \sum_{j=1}^n y_{i,j}^* \rho(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (5)$$

In the limit  $m, n \rightarrow \infty$ , we have

$$M_x = \int \int_D y \rho(x, y) dA \quad (6)$$

and

$$M_y = \int \int_D x \rho(x, y) dA \quad (7)$$

We define the center of the mass  $(\bar{x}, \bar{y})$ , which is a point,

$$\bar{x} = M_y/m, \quad \bar{y} = M_x/m \quad (8)$$

## 2 Surface Area

### 2.1 parameter domain is rectangle

First let's look at a smooth parametric surface  $S$ , which is given by the equation

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad (9)$$

and  $(u, v)$  is in a rectangle  $[a, b] \times [c, d]$ . Divide  $[a, b] \times [c, d]$  into  $m \times n$  small rectangles. We can see

$$A(S) = \sum_{i,j} A(S_{i,j}) \quad (10)$$

Here  $S_{i,j}$  is the image of the small rectangle  $(i, j)$ . Let's calculate (no, no estimate) the area of  $S_{i,j}$  when  $m$  and  $n$  are large enough. When  $m$  and  $n$  are very large, we can consider  $S_{i,j}$  as a parallelogram. We can see that

$$A(S_{i,j}) \approx |\vec{r}_u(u_{i,j}^*, v_{i,j}^*) \times \vec{r}_v(u_{i,j}^*, v_{i,j}^*)| \Delta u \Delta v \quad (11)$$

With  $A(S) = \sum_{i,j} A(S_{i,j})$ , letting  $m, n \rightarrow \infty$ , we obtain:

$$A(S) = \int_a^b \int_c^d |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| dv du \quad (12)$$

Example:

Find the surface area of a sphere of radius  $a$ .

1. First, we use the parametric representation:

$$x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi \quad (13)$$

2. The parameter domain is:

$$D = \{\phi, \theta | 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\} \quad (14)$$

3. Calculate  $\vec{r}_\phi$  and  $\vec{r}_\theta$

$$\vec{r}_\phi = \langle a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi \rangle \quad (15)$$

and

$$\vec{r}_\theta = \langle -a \sin \phi \cos \theta, a \sin \phi \sin \theta, 0 \rangle \quad (16)$$

4. Calculate  $|\vec{r}_\phi \times \vec{r}_\theta|$ :

$$|\vec{r}_\phi \times \vec{r}_\theta| = a^2 \sin \phi \quad (17)$$

5. At last, we have:

$$A = \int_0^{2\pi} \int_0^\pi a^2 \sin \phi d\phi d\theta = \int_0^{2\pi} 2a^2 d\theta = 4\pi a^2 \quad (18)$$

## 2.2 general parameter domain

As you can guess if the parameter domain is  $D$ , which is not rectangle, we have

$$A(S) = \int \int_D |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| dA \quad (19)$$

If the parameter domain is type I, we have the following formula,

$$A(S) = \int_a^b \int_{g_1(u)}^{g_2(u)} |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| dv du \quad (20)$$

If the the parameter domain is type II ?

## 2.3 Surface Area of Graph

Graph:  $x, y$  are the parameters.

$$x = x, \quad y = y, \quad z = f(x, y) \quad (21)$$

or

$$x = u, \quad y = v, \quad z = f(u, v) \quad (22)$$

So we have:  $\vec{r}_x = \langle 1, 0, f_x(x, y) \rangle$  and  $\vec{r}_y = \langle 0, 1, f_y(x, y) \rangle$ . Therefore,

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + f_x^2 + f_y^2} \quad (23)$$

Example:

Find the area of the part of the paraboloid  $z = x^2 + y^2$  that lies under the plane  $z = 9$ .

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2} dA \quad (24)$$

What is  $D$ ? what is  $\sqrt{1 + f_x^2 + f_y^2}$ ?

$$A(S) = \iint_{x^2+y^2 \leq 9} \sqrt{1 + 4x^2 + 4y^2} dA \quad (25)$$

Converting to polar coordinates, we obtain:

$$A(S) = \int_0^{2\pi} \int_0^3 r \sqrt{1 + 4r^2} dr d\theta = \frac{\pi}{6} (37^{3/2} - 1) \quad (26)$$

### 3 Triple integrals

We consider a function  $f$  of **three variables** defined on a closed box:

$$B = [a, b] \times [c, d] \times [e, f] \quad (27)$$

The first step is to divide the box  $R$  into small boxes. Let's cut  $R$  into  $m \times n \times l$  rectangles, each with volume

$$\Delta V = \Delta x \Delta y \Delta z \quad (28)$$

Then choose one sample point  $P_{i,j,k} = (x_{i,j,k}^*, y_{i,j,k}^*, z_{i,j,k}^*)$ . Triple Riemann sum is:

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l f(x_{i,j,k}^*, y_{i,j,k}^*, z_{i,j,k}^*) \Delta V \quad (29)$$

Triple Riemann integral is the limit when you take  $m, n, l \rightarrow \infty$ , i.e.,

$$\iiint_B f(x, y, z) dV = \lim_{n,m,l \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l f(x_{i,j,k}^*, y_{i,j,k}^*, z_{i,j,k}^*) \Delta V \quad (30)$$

and we can calculate it:

$$\iiint_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz \quad (31)$$

General Case: if  $f$  is defined on  $E$  and  $E \subset B$ , we have

$$\iiint_E f(x, y, z) dV = \iiint_E F(x, y, z) dV \quad (32)$$

what is  $F$ ?

what is the meaning of

$$\iiint_E 1 dV \quad (33)$$

A solid region  $E$  is said to be of **type one** if it lies between two graphs:

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (34)$$

Think about the cylinder and hemisphere.

We have the following result:

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (35)$$

Remember?  $3 = 2 + 1$ . If  $D$  is type 1, we have

$$\iiint_E f(x, y, z) dV = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dy \right] dx \quad (36)$$

**Note:** The range of the variable inside depends on the variable outside.

Example:

Evaluate  $\iiint_E z dV$ , where  $E$  is the solid tetrahedron bounded by the four planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and  $x + y + z = 1$ .

What is  $f$ ? what is  $E$ ?

$$\iiint_E z dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx = \frac{1}{24} \quad (37)$$

Switch  $x, y, z$ , we have the formula when  $E$  is type 2 or type 3.

**Mass:** If the density function of a solid object that occupies the region  $E$  is  $\rho(x, y, z)$ , in units of mass per unit volume, at any given  $(x, y, z)$ , then mass is :

$$m = \iiint_E \rho(x, y, z) dV \quad (38)$$

The center of mass  $(\bar{x}, \bar{y}, \bar{z})$  is:

$$\bar{x} = \frac{\int \int \int_E x \rho(x, y, z) dV}{\int \int \int_E \rho(x, y, z) dV} \quad (39)$$

so as  $\bar{y}, \bar{z}$ .

Example: Evaluate  $\int \int \int_E \sqrt{x^2 + y^2} dV$ , where  $E$  is the region bounded the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ . We have:

$$\int \int \int_E \sqrt{x^2 + y^2} dV = \int_2^4 \int_{x^2}^4 \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} dz dy dx \quad (40)$$

but it is very hard to do the calculation. On the other hand,

$$\begin{aligned} \int \int \int_E \sqrt{x^2 + y^2} dV &= \int \int_{x^2+z^2 \leq 4} \int_{x^2+z^2}^4 \sqrt{x^2 + z^2} dy dz dx \quad (41) \\ &= \int \int_{x^2+z^2 \leq 4} (4 - x^2 - z^2) \sqrt{x^2 + z^2} dz dx \end{aligned}$$

sssssshh hhug Using polar coordinates in  $xz$  plane:

$$\int \int_{x^2+z^2 \leq 4} (4 - x^2 - z^2) \sqrt{x^2 + z^2} dz dx = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = 128\pi/15 \quad (42)$$

# Note 19, Math21A

J. Yin

April 29, 2009

Bell's palsy lasts a few weeks.

## 1 Triple integrals

We consider a function  $f$  of **three variables** defined on a closed box:

$$B = [a, b] \times [c, d] \times [e, f] \quad (1)$$

The first step is to divide the box  $R$  into small boxes. Let's cut  $R$  into  $m \times n \times l$  rectangles, each with volume

$$\Delta V = \Delta x \Delta y \Delta z \quad (2)$$

. Then choose one sample point  $P_{i,j,k} = (x_{i,j,k}^*, y_{i,j,k}^*, z_{i,j,k}^*)$ . Triple Riemann sum is:

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l f(x_{i,j,k}^*, y_{i,j,k}^*, z_{i,j,k}^*) \Delta V \quad (3)$$

Triple Riemann integral is the limit when you take  $m, n, l \rightarrow \infty$ , i.e.,

$$\iiint_B f(x, y, z) dV = \lim_{n,m,l \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l f(x_{i,j,k}^*, y_{i,j,k}^*, z_{i,j,k}^*) \Delta V \quad (4)$$

and we can calculate it:

$$\iiint_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz \quad (5)$$

General Case: if  $f$  is defined on  $E$  and  $E \subset B$ , we have

$$\iiint_E f(x, y, z) dV = \iiint_B F(x, y, z) dV \quad (6)$$

what is  $F$ ?

what is the meaning of

$$\int \int \int_E 1 dV \quad (7)$$

A solid region  $E$  is said to be of **type one** if it lies between two graphs:

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (8)$$

Think about the cylinder and hemisphere.

cylinder:  $D = \{(x, y) : x^2 + y^2 \leq 1\}, 0 \leq z \leq 1$ .

Hemisphere:  $D = \{(x, y) : x^2 + y^2 \leq 1\}, 0 \leq z \leq \sqrt{1 - x^2 - y^2}$

We have the following result:

$$\int \int \int_E f(x, y, z) dV = \int \int_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (9)$$

Remember?  $3 = 2 + 1$ . If  $D$  is type 1, we have

$$\int \int \int_E f(x, y, z) dV = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dy \right] dx \quad (10)$$

**Note:** The range of the variable inside depends on the variable outside.

Example:

Evaluate  $\int \int \int_E z dV$ , where  $E$  is the solid tetrahedron bounded by the four planes  $x = 0, y = 0, z = 0$  and  $x + y + z = 1$ .

What is  $f$ ? what is  $E$ ?

$$\int \int \int_E z dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx = \frac{1}{24} \quad (11)$$

Switch  $x, y, z$ , we have the formula when  $E$  is type 2 or type 3.

**Mass:** If the density function of a solid object that occupies the region  $E$  is  $\rho(x, y, z)$ , in units of mass per unit volume, at any given  $(x, y, z)$ , then mass is :

$$m = \int \int \int_E \rho(x, y, z) dV \quad (12)$$

The center of mass  $(\bar{x}, \bar{y}, \bar{z})$  is:

$$\bar{x} = \frac{\int \int \int_E x \rho(x, y, z) dV}{\int \int \int_E \rho(x, y, z) dV} \quad (13)$$

so as  $\bar{y}$ ,  $\bar{z}$ .

Example: Evaluate  $\int \int \int_E \sqrt{x^2 + z^2} dV$ , where  $E$  is the region bounded the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ . We have:

$$\int \int \int_E \sqrt{x^2 + z^2} dV = \int_2^4 \int_{x^2}^4 \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} dz dy dx \quad (14)$$

but it is very hard to do the calculation. On the other hand, we can see that this question is equivalent to: Evaluate  $\int \int \int_E \sqrt{x^2 + y^2} dV$ , where  $E$  is the region bounded the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$ .

$$\begin{aligned} \int \int \int_E \sqrt{x^2 + y^2} dV &= \int \int_{x^2+y^2 \leq 4} \int_{x^2+y^2}^4 \sqrt{x^2 + y^2} dz dy dx \quad (15) \\ &= \int \int_{x^2+y^2 \leq 4} (4 - x^2 - y^2) \sqrt{x^2 + y^2} dy dx \end{aligned}$$

Using polar coordinates in  $xy$  plane:

$$\int \int_{x^2+y^2 \leq 4} (4 - x^2 - y^2) \sqrt{x^2 + y^2} dz dx = \int_0^{2\pi} \int_0^2 (4 - r^2) r^2 dr d\theta = 128\pi/15 \quad (16)$$

## 2 Triple Integrals in Cylindrical and Spherical Coordinates

### 2.1 Cylindrical Coordinates

Recall the Cylindrical Coordinates  $r, \theta, z$ . When should we use Cylindrical Coordinates?

When the Domain  $E$  can be described as

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (17)$$

where  $D$  is given in polar coordinates by

$$D = \{(r, \theta), \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\} \quad (18)$$

With property (17), we have

$$\int \int \int_E f(x, y, z) dV = \int \int_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (19)$$

Together with (18), we have

$$\int \int \int_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta \quad (20)$$

## 2.2 Spherical Coordinates

Recall Spherical Coordinates:  $\rho, \theta, \phi$ ,

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi \quad (21)$$

When should we use Spherical Coordinates? When  $E$  can be easily described by Spherical Coordinates, e.g.

$$E = \{(\rho, \theta, \phi) : a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\} \quad (22)$$

We have the following important formula:

$$\int \int \int_E f(x, y, z) dV = \int_c^d \int_{\alpha}^{\beta} \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi \quad (23)$$

Or a little bit complicated

$$E = \{(\rho, \theta, \phi) : \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\} \quad (24)$$

we have

$$\int \int \int_E f(x, y, z) dV = \int_c^d \int_{\alpha}^{\beta} \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi \quad (25)$$

Example:

Evaluate  $\int \int \int_B 1 dV$ , where  $B$  is the unit ball

$$\begin{aligned} \int \int \int_B 1 dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \int_0^{\pi} \int_0^{2\pi} \sin \phi \frac{1}{3} d\phi d\theta = ?? \end{aligned} \quad (26)$$

Example:

Find the volume of the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$ .

First rewrite these two equations, we obtain that solid lie above

$$\rho = \cos \phi \text{ and } \phi = \pi/4 \quad (27)$$

We rewrite the domain  $E$  as

$$E = \{(\rho; \theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4, 0 \leq \rho \leq \cos \phi\} \quad (28)$$

So we have

$$\begin{aligned} \iiint_E 1dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{\pi} \int_0^{2\pi} \sin \phi \frac{\cos \phi^3}{3} d\phi d\theta = \frac{\pi}{8} \end{aligned} \quad (29)$$

### 3 Vector Fields

Definition: Let  $D$  be a set in  $\mathbb{R}^2$ , A **vector field** on  $\mathbb{R}^2$  is a function  $F$  that assign to each point  $(x, y)$  in  $D$  a two dimensional vector  $F(x, y)$ .

Component functions:

$$F(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} \quad (30)$$

i.e.  $F = P\vec{i} + Q\vec{j}$ . How to define vector field on  $\mathbb{R}^3$ ?

$$F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k} \quad (31)$$

$P, Q, R$  are scalar fields.

We can draw (roughly) the vector field as follows. Pick some points in  $D$ , for  $(x, y)$ , we draw a vector  $F(x, y)$  with the initial (start ) point  $(x, y)$ .

Example:

Sketch the vector field  $F(x, y)$ :

$$F(x, y) = -y\vec{i} + x\vec{j} \quad (32)$$

#### 3.1 Gradient Fields

$f$  is a scalar field (ordinary function) of two variables, recall gradient fields

$$\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j} \quad (33)$$

Three variable??

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k} \quad (34)$$

Recall  $\nabla f(x, y)$  perpendicular to the contour lines. Examples.

# Note 20, Math21A

J. Yin

April 29, 2009

Bell's palsy lasts a few weeks.

## 1 Vector Fields

Definition: Let  $D$  be a set in  $\mathbb{R}^2$ , A **vector field** on  $\mathbb{R}^2$  is a function  $F$  that assign to each point  $(x, y)$  in  $D$  a two dimensional vector  $F(x, y)$ .

Component functions:

$$F(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} \quad (1)$$

i.e.  $F = P\vec{i} + Q\vec{j}$ . How to define vector field on  $\mathbb{R}^3$ ?

$$F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k} \quad (2)$$

$P, Q, R$  are scalar fields.

### 1.1 Gradient Fields

$f$  is a scalar field (ordinary function) of two variables, recall gradient fields

$$\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j} \quad (3)$$

Three variable??

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k} \quad (4)$$

Example: weather map.

## 2 Line Integrals

### 2.1 Plane curve

We start with a plane curve  $C$  given by the parametric equation

$$x = x(t), y = y(t), a \leq t \leq b \quad (5)$$

For example, circle.

Let's define

$$\int_C f(x, y) ds \quad (6)$$

Note, the  $C$  means the curve, the  $s$  is not a parameter.  $ds$  is kinda like  $dA$ ,  $dV$ .

Dividing  $C$  into small pieces, let's say  $m$  pieces, pick one point  $(x_i^*, y_i^*)$  in each pieces. Then we can calculate the Riemann sum:

$$\sum_n f(x_i^*, y_i^*) \Delta s_i \quad (7)$$

The limit of Riemann sum is Riemann integral.

$$\int_C f(x, y) ds = \lim \sum_n f(x_i^*, y_i^*) \Delta s_i \quad (8)$$

Remember the following formula,

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (9)$$

Example:  $f(x, y) = xy$ ,  $C : x(t) = \sin t$ ,  $y(t) = \cos t$ ,  $0 \leq t \leq \pi$

$$\int_C f(x, y) ds = \int_0^\pi \sin t \cos t dt = 0 \quad (10)$$

### 2.2 Curve in Space

$$x = x(t), y = y(t), z = z(t) \quad a \leq t \leq b \quad (11)$$

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt \end{aligned} \quad (12)$$

### 2.3 Line Integrals of Vector Fields

Let  $F$  be continuous force field on  $R^3$ ,

$$\vec{F}(x, y, z) = P\vec{i} + Q\vec{j} + R\vec{k} \quad (13)$$

For example: electric force field. If a particle (object) moves along curve  $C$ :

$$x = x(t), y = y(t), z = z(t) \quad a \leq t \leq b \quad (14)$$

Question: The work done by the force  $\vec{F}$  on this particle from  $t = a$  to  $t = b$ .

Dividing the curve into small pieces, in one small piece, work done by the force is about

$$\vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*) \Delta s_i \quad (15)$$

$$W \approx \sum_i \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*) \Delta s_i \quad (16)$$

In the limit:

$$W = \int_C \vec{F} \cdot \vec{T} ds \quad (17)$$

Here  $\vec{T}$  is the unit tangent vector. We have

$$\begin{aligned} W &= \int_C \vec{F} \cdot \vec{T} ds = \int \left[ \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \right] |\vec{r}'(t)| dt \quad (18) \\ &= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \end{aligned}$$

Usually, we write it as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \quad (19)$$

Don't forget the direction of  $C$  and

$$\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r} \quad (20)$$

Example Calculate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F}(x, y, z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$  and  $C$  is the twisted cubic given by

$$x = t, y = t^2, z = t^3 \quad (21)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = 27/28 \quad (22)$$

Worksheet and hints.

### 3 Fundamental Theorem for Line Integrals

Do you remember this formula? It is called Fundamental Theorem of Calculus(big title)

$$\int_a^b F'(x)dx = F(b) - F(a) \quad (23)$$

The Fundamental Theorem for Line Integrals is following:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \quad (24)$$

or

$$\int_C \nabla f \cdot d\vec{r} = f(x_f, y_f, z_f) - f(x_i, y_i, z_i) \quad (25)$$

1. There exists a  $f$ , s.t.  $\vec{F} = \nabla f$  ( $\vec{F}$  is **conservative**)
2. If  $C$  is a closed path, i.e.,  $(x_f, y_f, z_f) = (x_i, y_i, z_i)$

$$\int_C \vec{F} \cdot d\vec{r} = 0 \quad (26)$$

3. Independence of Path

4.  $f$  is well defined:

$$f(x, y) = \int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r} \quad (27)$$

There is an easy way to check whether  $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$  is conservative. If  $P, Q$  has continuous first-order partial derivatives on a domain  $D$  and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (28)$$

holds in  $D$ , we have  $\vec{F}(x, y)$  is conservative.

It can be understood in this way

$$\vec{F} = \nabla f \Rightarrow P = f_x, Q = f_y \quad (29)$$

so

$$\frac{\partial P}{\partial y} = f_{xy} = \frac{\partial Q}{\partial x} \quad (30)$$

**But, But** this is true only for a special type of  $D$ , which is called **simply-connected region**. Intuitively speaking, simply-connected region contains no holes and can't consist of two separated pieces.

Hint for question 29-32, open, connect.

# Note 21, Math21A

J. Yin

April 29, 2009

## 1 Review: Integrals

The basic idea is dividing (line segment, rectangle, box) into many small pieces, then calculating the Riemann sum. The limit of Riemann sum is Riemann Integral.

### 1.1 2C

For scalar field  $f(x, y, z)$

$$x = x(t), y = y(t), z = z(t) \quad a \leq t \leq b \quad (1)$$

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (2) \\ &= \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt \end{aligned}$$

For vector fields  $\vec{F}(x, y, z)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \quad (3)$$

### 1.2 3C

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy \quad (4)$$

$$D \equiv \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (5)$$

$$\int \int_D f(x, y) dA = \int \int_R F(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (6)$$

### 1.3 4C

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (7)$$

### 1.4 5C

For general surface.

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \quad (8)$$

$$A(S) = \iint_D |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| dA \quad (9)$$

For graph,  $(u, v)$  is  $(x, y)$

$$(x, y, f(x, y)) \in \mathbb{R}^3, z = f(x, y) \quad (10)$$

$$A(G) = \iint_D \sqrt{1 + f_x^2 + f_y^2} dA \quad (11)$$

### 1.5 6C

$$\iiint_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz \quad (12)$$

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (13)$$

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (14)$$

Then it is just a question of 3C.

$$D \equiv \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (15)$$

$$\iiint_E f(x, y, z) dV = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dy \right] dx \quad (16)$$

### 1.6 7C

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (17)$$

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (18)$$

Then it is just a question of 4C. (Do not forget  $r$ )

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta \quad (19)$$

## 1.7 8C

$$E = \{(\rho, \theta, \phi) : \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\} \quad (20)$$

we have

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi \quad (21)$$

## 1.8 Which part is most difficult

Describe  $E$  and  $D$ .

## 1.9 Some Properties

$$\int_X f + g dY = \int_X f dY + \int_X g dY \quad (22)$$

$$\int_X c f dY = c \int_X f dY \quad (23)$$

$$\int_{X_1+X_2} f dY = \int_{X_1} f dY + \int_{X_2} f dY \quad (24)$$

## 1.10 Some definitions

Mass, center of mass.

## 2 Green's theorem

We start with reviewing fundamental formula:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \quad (25)$$

So for closed  $C$ , we have  $\int_C \nabla f \cdot d\vec{r} = 0$ . How about general  $\vec{F} = P\vec{i} + Q\vec{j}$ ? what is  $\int_C \vec{F} \cdot d\vec{r}$ ?

First, don't forget the curve has direction. Let's define positive orientation closed simple curve  $C$ :  $\vec{r}(t), a \leq t \leq b$ .

1. closed:  $\vec{r}(a) = \vec{r}(b)$
2. simple: for any  $a \leq c \neq d \leq b$ ,  $\vec{r}(c) \neq \vec{r}(d)$

3. positive orientation: counterclockwise.

**Green's theorem:**

Let  $C$  be a positively oriented, piecewise-smooth, simple closed curve in the plane and let  $D$  be the region bounded by  $C$ .  $\vec{F} = P\vec{i} + Q\vec{j}$ . If  $P$  and  $Q$  have continuous partial derivatives on an open region that contain  $D$ , then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (26)$$

If  $C$  is negatively oriented? Notations. Add a  $O$  to  $\int$ , it looks like  $\oint$ .

Example: Evaluate  $\oint x^4 dx + xydy$ , where  $C$  is the triangular curve consisting of line segments from  $(0,0)$  to  $(1,0)$ , from  $(1,0)$  to  $(0,1)$  and from  $(0,1)$  to  $(0,0)$

$$\oint x^4 dx + xydy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (y-0) dy dx = \frac{1}{6} \quad (27)$$

If the region bounded by  $C$  is a disk or semi-disk, we can use polar coordinates.

For a  $D$ , we can use green's theorem to calculate the area of  $D$ . Choose  $P$  and  $Q$  s.t.  $\left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 1$

$$\iint_D 1dA = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy \quad (28)$$

We can choose:

1.  $P(x, y) = 0, Q(x, y) = x$
2.  $P(x, y) = -y, Q(x, y) = 0$
3.  $P(x, y) = -y/2, Q(x, y) = x/2$

So, we have

$$A(D) = \oint_C xdy = - \oint_C ydx = \frac{1}{2} \oint_C (xdy - ydx) \quad (29)$$

Example: Find the area enclosed by the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (30)$$

The closed curve for the boundary is  $x = a \cos t, y = b \sin t$ . (Check the orientation)

$$A = \frac{1}{2} \oint_C (xdy - ydx) = \frac{ab}{2} \int_0^{2\pi} dt = \pi ab \quad (31)$$

There is a relative easy way.  $xdy = a \cos t db \sin t = ab(\cos t)^2$ .

### 3 Curl and Divergence

For vector field  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$  (must be three dimensional), Curl of  $\vec{F}$  is defined as follows

$$\text{curl } \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \quad (32)$$

How to remember this?

**Theorem:**

If  $f$  is a function of three variables that has continuous secondary order partial derivatives, then

$$\text{curl } \nabla f = \vec{0} \quad (33)$$

Since a conservative vector field is one for which  $\vec{F} = \nabla f$ , then with this theorem, we know that if  $\vec{F}$  is conservative, then  $\text{curl } \vec{F} = 0$ .

# Note 22, Math21A

J. Yin

April 23, 2009

## 1 Curl and Divergence

### 1.1 Curl

For vector field  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$  (must be three dimensional), Curl of  $\vec{F}$  is defined as follows

$$\text{curl } \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \quad (1)$$

How to remember this? Do you remember  $\nabla$ ?

$$\nabla f = \langle f_x, f_y, f_z \rangle \quad (2)$$

We can write  $\nabla$  as a vector as follows:

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \quad (3)$$

Then we can remember the definition as

$$\text{curl } \vec{F} = \nabla \times \vec{F} \quad (4)$$

1. For the students who are familiar with determinant. We know  $\vec{a} = \langle a_1, a_2, a_3 \rangle$ ,  $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (5)$$

Replace  $\vec{a}$  with  $\nabla$  and  $\vec{b}$  with  $\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad (6)$$

Let's calculate this determinant. It is equal to Curl  $\vec{F}$ .

2. It can also be remembered by the following method (the way I like to use)

$$1 - 2 - 3, 2 - 3 - 1, 3 - 1 - 2 \quad (7)$$

**Theorem 1:**

If  $f$  is a function of three variables that has continuous secondary order partial derivatives, then

$$\text{curl } \nabla f = \vec{0} \quad (8)$$

Since a conservative vector field is one for which  $\vec{F} = \nabla f$ , then with this theorem, we know that if  $\vec{F}$  is conservative, then  $\text{curl } \vec{F} = 0$ .

On the other hand, we have the following theorem.

**Theorem 2:**

If  $\vec{F}$  is a vector field defined on all of  $\mathbb{R}^3$  whose component functions have continuous partial derivatives and  $\text{Curl } \vec{F} = 0$ , then  $\vec{F}$  is conservative vector field.

The next question should be: if we use  $\vec{F}$  is conservative then how can we find the  $f$  s.t.  $\nabla f = \vec{F}$ .

Example:

Show that  $\vec{F}(x, y, z) = y^2 z^3 \vec{i} + 2xyz^3 \vec{j} + 3xy^2 z^2 \vec{k}$  is conservative. Find a function  $f$  s.t.  $\nabla f = \vec{F}$ .

First, we calculate  $\nabla \times \vec{F}$ :

$$\nabla \times \vec{F} = (6xyz^2 - 6xyz^2)\vec{i} + (3y^2 z^2 - 3y^2 z^2)\vec{j} + (2yz^3 - 2yz^3)\vec{k} = \vec{0} \quad (9)$$

We need to find a  $f$  s.t

$$f_x = y^2 z^3 \quad (10)$$

So,  $f(x, y, z) = xy^2 z^3 + g(y, z)$ . Again with  $f_y = 2xyz^3$ , we have  $g_y(y, z) = 0$ , which implies  $g(y, z) = h(z)$ . At last, with  $f_z = 3xy^2 z^2$ , we have  $h(z)$  is equal to constant. So  $f$  is equal to  $xy^2 z^3 + \text{constant}$ .

## 1.2 Divergence

Definition: For vector field  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\text{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad (11)$$

If we consider  $\nabla$  as vector, we can write it as

$$\text{div}F = \nabla \cdot F \quad (12)$$

Compare  $\text{grad } f$ ,  $\text{curl } \vec{F}$ ,  $\text{div } \vec{F}$ .

By the definition, we have the following theorem.

**Theorem 3:**

If  $\vec{F}$  is a vector field defined on all of  $\mathbb{R}^3$  whose component functions have continuous partial derivatives, then

$$\text{div} (\nabla \times \vec{F}) = 0 \quad (13)$$

With this theorem, for fixed  $\vec{F}$ , we can answer whether there exist  $\vec{G}$  s.t.  $F = \vec{G}$ .

Give a hint for question 30.

## 2 Surface Integrals

New kind of integrals? Right, actually two kinds of new integrals.

### 2.1 Surface Integral of Scalar field

Definition:

Let  $\vec{r}(u, v)$ ,  $(u, v) \in D$  be the surface and  $f(x, y, z)$  is the integrand. Dividing the surface into small pieces (actually we divide the  $D$  in the parametric space into small pieces) and pick one point on the each piece, calculating the Riemann sum

$$\sum_{i=1}^m \sum_{j=1}^n f(P_{i,j}^*) \Delta S_{i,j} \quad (14)$$

where  $S_{i,j} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$ . In the limit  $m, n \rightarrow \infty$ , we have

$$\int \int_S f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{i,j}^*) \Delta S_{i,j} = \int \int_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA \quad (15)$$

Compare this formula with the the formula for surface area.

For graph??  $(x, y, z) = (x, y, g(x, y))$ .

$$\int \int_S f(x, y, z) dS = \int \int_D f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} dA \quad (16)$$

Mass:

$$m = \int \int_S \rho(x, y, z) dS \quad (17)$$

and center of mass is  $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{\int_S x\rho(x, y, z)dS}{\int_S \rho(x, y, z)dS} \quad (18)$$

so as  $\bar{y}, \bar{z}$ .

## 2.2 Surface Integral of vector field

Recall the line integral of vector field.

$$\int_C \vec{F} \cdot \vec{T} ds \quad (19)$$

It is equal to line integral of  $\vec{F} \cdot \vec{T}$ . The Surface Integral of vector field is kind like this.

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int_S \vec{F} \cdot \vec{n} dS \quad (20)$$

Here  $\vec{n}$  is the normal vector. (YOU MUST PICK ONE of TWO).

$$\vec{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad (21)$$

So we can see

$$\int \int_S \vec{F} \cdot d\vec{S} = \pm \int \int_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA \quad (22)$$

In the case of a surface  $S$  given by a graph  $z = g(x, y)$ , we can think of  $x$  and  $y$  as parameters and we obtain:

$$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) \quad (23)$$

So we have

$$\int \int_S \vec{F} \cdot d\vec{S} = \pm \int \int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad (24)$$

Example:  $S$  is the surface of sphere with radius  $a$  and  $\vec{F}$  is  $\langle z, y, x \rangle$ .  $\int \int_S \vec{F} \cdot d\vec{S}$ ?

Surface of sphere, we use

$$\vec{r} = \langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \theta \rangle, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi \quad (25)$$

we have:

$$\vec{r}_\phi \times \vec{r}_\theta = a \sin \phi \vec{r} \quad (26)$$

is it outward or inward? If we pick the outward  $S$ , we have

$$\int \int_D \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) dA = \int_0^{2\pi} \int_0^\pi (2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta) d\phi d\theta \quad (27)$$

It is equal to  $4\pi/3$ .

What is meaning of  $\int \int_S \vec{F} \cdot \vec{n} dS$ ?

Let  $S$  is a oriented surface with unit normal vector  $\vec{n}$ . and image a fluid with density  $\rho(x, y, z)$  and velocity field  $v(x, y, z)$  flowing through  $S$ . (Think of  $S$  as an imaginary surface that does not impede the fluid flow, like a fishing net across a stream) Then the rate of flow (mass per unit time) per unit area is  $\rho\vec{v}$ . For a small piece  $S_{i,j}$ , we can estimate that the mass of fluid crossing  $S_{i,j}$  per unit time is about

$$\rho\vec{v} \cdot nA(S_{i,j}) \quad (28)$$

The total mass of fluid crossing the surface is

$$\sum_{i,j} \rho\vec{v} \cdot nA(S_{i,j}) \quad (29)$$

So in the limit, we have

$$\int \int_S \rho\vec{v} \cdot ndS \quad (30)$$

If we write  $\vec{F} = \rho\vec{v}$ , we get  $\int \int_S \vec{F} \cdot ndS$  is the flux of  $F$  across  $S$ .

It is a very useful integral in physics.

# Note 23, Math21A

J. Yin

April 28, 2009

## 1 Stokes' Theorem

### 1.1 Stoke who?

Do you remember we have a theorem (green's theorem) tells that line integral of vector fields (on plane) is equal to some double integrals on the plane.

How about a curve in space? First let's look at the surface with only one side.

**Stokes' Theorem** Let  $S$  be an oriented piecewise-smooth surface that is bounded by closed, piecewise-smooth boundary curve  $C$  with positive orientation. Let  $\vec{F}$  be a vector field whose components have continuous partial derivatives on an open region in  $\mathbb{R}^3$  that contains  $S$ . Then

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_S \nabla \vec{F} \cdot d\vec{S} \quad (1)$$

Since

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds, \quad \int \int_S \nabla \vec{F} \cdot d\vec{S} = \int \int_S \nabla F \cdot \vec{n} dS \quad (2)$$

Stokes' theorem says that line integral around the boundary curve of  $S$  of the tangential component of  $\vec{F}$  is equal to the surface integral of the normal component of Curl of  $\vec{F}$ .

What is relation between  $\vec{T}$  and  $\vec{n}$ ? If you walk in the positive direction around  $C$  with your head pointing in the direction of  $\vec{n}$ , the surface will always be on your left.

It holds for **Any surfaces** with the same boundary.

## 1.2 Examples

Example 1: Evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = \langle -y^2, x, z^2 \rangle$  and  $C$  is curve of intersection of plane  $y + z = 2$  and cylinder  $x^2 + y^2 = 1$ . (oriented  $C$  to be counterclockwise when view from above)

Solution: Although  $\int_C \vec{F} \cdot d\vec{r}$  could be evaluated directly, it's easier to use Stokes's theorem. We first calculate  $\text{Curl} \vec{F}$ .

$$\text{Curl} \vec{F} = (1 + 2y)\vec{k} \quad (3)$$

There are many surfaces we can choose, but we like to choose the most convenient one, which is the one in the plane  $y + z = 2$ . This is graph:

$$\{(x, y, 2 - y) : x^2 + y^2 \leq 1\} \quad (4)$$

Now, we need to calculate the surface integral of vector fields  $(1 + 2y)\vec{k}$ . Recall the formula: So we can see

$$\iint_S \nabla \vec{F} \cdot d\vec{S} = \pm \iint_D \nabla \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA \quad (5)$$

In the case of a surface  $S$  given by a graph  $z = g(x, y)$ , we can think of  $x$  and  $y$  as parameters and we obtain: (Here  $\nabla \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ )

$$\nabla \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) \quad (6)$$

So we have

$$\iint_S \nabla \vec{F} \cdot d\vec{S} = \pm \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad (7)$$

In our case,  $P = Q = 0$ ,  $D$  is disk.

$$\iint_D (1 + 2y) dA = \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r dr d\theta = \pi \quad (8)$$

Example 2: Compute the integral  $\int_S \text{Curl} F \cdot d\vec{S}$ , where  $F = \langle xz, yz, xy \rangle$  and  $S$  is the part of the sphere  $x^2 + y^2 + z^2 = 4$  that lies inside the cylinder  $x^2 + y^2 = 1$  and above the  $x - y$  plane.

Solution: First we can find this curve (the boundary of the surface):

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + \sqrt{3}\vec{k} \quad (9)$$

With Stokes' theorem

$$\int \int_S \nabla \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \quad (10)$$

Just inset (9) into (10).

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t) dt = 0 \quad (11)$$

Again, in general, if  $S_1$  and  $S_2$  are oriented surfaces with the same oriented boundary curve  $C$ , then

$$\int \int_{S_1} \nabla \vec{F} \cdot d\vec{S} = \int \int_{S_2} \nabla \vec{F} \cdot d\vec{S} \quad (12)$$

## 2 Divergence Theorem

### 2.1 Divergence is not someone's name

We have seen that the integral on closed curve  $C$  is equal to some other integral on the surface whose boundary  $C$ . Now we are going to see the integral on closed surface  $S$  is equal to some other integral on the solid region  $E$  whose boundary  $S$ .

**Divergence theorem:**

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_E \text{div} F dV \quad (13)$$

Is there anything missing?  $S$  is positive(outward) orientation.

### 2.2 Examples

Example 1: Find the flux of the vector field  $\vec{F} = \langle z, y, x \rangle$  over the unit sphere  $x^2 + y^2 + z^2 = 1$ .

Solution: First we calculate  $\text{div} \vec{F} = 1$ . So

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_{Ball} \text{div} \vec{F} dV = 4\pi/3 \quad (14)$$

### 2.3 Source and Sink

Think  $\vec{F}$  as the fluid flow. We know

$$\int \int \int_{Ball} \text{div} \vec{F} = \int \int_S \vec{F} \cdot d\vec{s} \quad (15)$$

When the ball is very small, we have

$$\text{div} \vec{F}(P_0) \approx \frac{1}{V(Ball)} \int \int_S \vec{F} \cdot d\vec{s} \quad (16)$$

So the  $\text{div} \vec{F}(P_0)$  is the net rate of outward flux per unit volume at  $P_0$ . If  $\text{div} \vec{F}(P_0) > 0$ , the net flux is outward near P and P is called **source**. If  $\text{div} \vec{F}(P_0) < 0$ , the net flux is inward near P and P is called **sink**.

# Note for Final exam, Math21A

J. Yin

April 30, 2009

## 0.1 EXAMS COMING..HOW??

Every College Students Least Favorite Time of the Year.

1. Redo the homework
2. Practice the old exams
3. Review the notes and questions we used in class.
4. Make a list of all the concepts and types of questions you have seen.

There are some easy ways:

1. Call the police and say you placed a bomb in the classroom so they have to cancel the exam.
2. Write the whole book on your boots(Find a big one)/ For boys, get a big tie.
3. Hack my computer.
4. Believe me, Peter don't remember the face of everyone.

For each subtitle, we need to know what is that, how to describe it, how to calculate it, what kind of questions we have seen about it.

## 1 Before the first midterm exam

### 1.1 Vector

Vector, Dot Product, Cross Product.

## 1.2 Coordinate System

1. Plane: Rectangular Coordinate System, Polar Coordinate System
2. Space: Rectangular Coordinate System, Cylindrical Coordinate System and Spherical Coordinate System

## 1.3 Plane and line

Equations for plane or line. Distance between different objects.

## 1.4 Surface and Graph

Parameterization of surface. what's Graph? Do you remember the quadrics in Table two Pg682.

## 1.5 Space Curve

Parameterization of curve, tangent vector, normal vector, binormal vector, normal plane, binormal plane.

## 1.6 Partial derivative

$f_x, f_y, f_{xy}, f_{xx}$ . Check your calculation.

# 2 Between the first midterm and the second one

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## 2.1 Linear approximation

How can we estimate  $f(x, y, z)$  or  $f(x, y)$  with Linear approximation?

## 2.2 Directional derivatives and Gradient vector

$D_u f(x_0, y_0)$  is a number but  $\nabla f(x_0, y_0)$  is a vector. With gradient vector, we can easily obtain the tangent line in contour map and tangent plane for surface.

## 2.3 Maximum and Minimum

1. Critical point, local minimum point and local maximum point, saddle point. Second derivative test.

2. How can we find critical points on contour map and classify them?  
Circle and Cross.
3. How can we find the maximum or minimum in some region?

## 2.4 Chain rules and implicit question

Will not be harder than Hws

## 2.5 Lagrange Multipliers

Typical question: Find the maximum or minimum values of  $f$  subject to constraint  $g(x, y, z) = k$ . The method is clear. But it is easy to make mistake.

## 2.6 Fubini's Theorem

Sometimes you need to change the order of  $dx dy$

## 2.7 Double integrals in Polar coordinate

Don't forget  $r$ .

## 2.8 Mass and Center of Mass

$$\bar{x} = \frac{\int_D x \rho(x, y) dA}{\int_D \rho(x, y) dA} \quad (1)$$

## 3 Eleven Kinds of integrals

1. 1A: On line segment.
2. 1B: line integral of Scalar field: a function  $f$  and curve. Use the formula and get the correct answer. But sometimes the question does not give you the curve directly.
3. 1C: line integral of Scalar field: a vector function  $F$  and curve. Use the formula and get the correct answer. But don't forget the direction.
4. 2A: 2D- rectangular CS.  $dx dy$ . Fubini Theorem. Type one, type two.
5. 2B: 2D- Polar CS.  $dr d\theta$ .  $r$  there is a  $r$ .

6. 2C: Surface integral of Scalar fields: the special case is surface area, i.e,  $f = 1$ .

$$\int \int_D f(u, v) |\vec{r}_u \times \vec{r}_v| dA \quad (2)$$

$D$  is subset in  $u - v$  plane. But sometimes the question does not give you the surface directly

7. 2D: Special case the Surface is a graph

$$\int \int_D f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} dA \quad (3)$$

$D$  is the project of the graph on  $x - y$  plane.

8. 2E: Surface integral of vector fields:  $\vec{F} \cdot \vec{n}$ . Pick the orientation of the surface.

$$\int \int_D \vec{F}(u, v) \cdot (\vec{r}_u \times \vec{r}_v) dA \quad (4)$$

$D$  is subset in  $u - v$  plane.

9. 2F: Special case the Surface is a graph. If  $\vec{F} = \langle P, Q, R \rangle$

$$\int \int_D (-Pg_y - Qg_x + R) dA \quad (5)$$

$D$  is the project of the graph on  $x - y$  plane.

10. 3A: 3D- rectangular CS.  $dx dy dz$ . Fubini Theorem. Type one, type two, type three. Practice more.

$$\int \int \int_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz \quad (6)$$

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (7)$$

$$\int \int \int_E f(x, y, z) dV = \int \int_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (8)$$

Then it is just a question of 2A.

$$D \equiv \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (9)$$

$$\int \int \int_E f(x, y, z) dV = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dy \right] dx \quad (10)$$

You may realize many question only tells you the solid is bounded by what, what and what. (really hard)

11. 3B: 3D- Cylindrical CS.  $drd\theta dz$ .

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \quad (11)$$

$$\int \int \int_E f(x, y, z) dV = \int \int_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \quad (12)$$

Then it is just a question of 2B. (Do not forget  $r$ )

$$\int \int \int_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta \quad (13)$$

12. 3C: 3D- Spherical CS.  $d\rho d\theta d\phi$ .

$$E = \{(\rho, \theta, \phi) : \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\} \quad (14)$$

we have

$$\int \int \int_E f(x, y, z) dV = \int_c^d \int_{\alpha}^{\beta} \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi \quad (15)$$

Note: We did not study any 3D integrals of vector fields.

## 4 Higher-dimensional Version of the Fundamental Theorem of Calculus

$$\text{grad } f \equiv \nabla f, \text{Curl } \vec{F} \equiv \nabla \times \vec{F}, \text{div } \vec{F} \equiv \nabla \cdot \vec{F}$$

1. Fundamental Theorem of Calculus:

$$\int_a^b F'(x) dx = F(b) - F(a) \quad (16)$$

2. Fundamental Theorem of line integral:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(a)) - f(\vec{r}(b)) \quad (17)$$

3. Green's theorem  $\vec{F} = P\vec{i} + Q\vec{j}$  on  $x - y$  plane

$$\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C \vec{F} \cdot d\vec{r} \quad (18)$$

4. Stoke's theorem  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int \int_S \nabla \times \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \quad (19)$$

5. Divergence theorem  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int \int \int_E \nabla \cdot \vec{F} dV = \int \int_S \vec{F} \cdot d\vec{s} \quad (20)$$

How to check a vector field  $\vec{F}$  is conservative or not?  $\vec{F} = \nabla f$ ,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0, \vec{F} = P\vec{i} + Q\vec{j} \quad (21)$$

$$\nabla \times \vec{F} = 0 \quad (22)$$

How to check a vector field  $\vec{F} = \nabla \vec{G}$  or not?  $div \vec{F} = 0$

#### 4.1 Relation between different integrals

1. 1A knows everyone
2. 1B knows 1C ( $\vec{F} \cdot \vec{T}$ ) so as (2E, 2C) and (2F, 2D)
3. 3A knows 2A, 3B knows 2B.
4. 1C knows 2A, with green's theorem, only planar curve.
5. 1C knows 2E, with Stokes theorem.
6. 3A, 3B, 3C know each other, they know 2E, with divergence theorem.

You may find more.